HARVARD UNIVERSITY
Graduate School of Arts and Sciences

DISSERTATION ACCEPTANCE CERTIFICATE

The undersigned, appointed by the

Department of Mathematics

have examined a dissertation entitled

A monoidal structure for the Fukaya category

presented by **Aleksandar Subotic**

candidate for the degree of Doctor of Philosophy and hereby certify that it is worthy of acceptance.

Signature D Auroux
Typed name: Denis Auroux (M.I.T.)

Signature
Typed name: Clifford Taubes

Signature
Typed name: Shing-Tung Yau

Signature
Typed name:

Date: April 29, 2010

A Monoidal Structure for the Fukaya Category

A dissertation presented
by

Aleksandar Subotic

to
The Department of Mathematics
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in the subject of

Mathematics

Harvard University
Cambridge, Massachusetts
May 2010

UMI Number: 3415373

All rights reserved

INFORMATION TO ALL USERS
The quality of this reproduction is dependent upon the quality of the copy submitted.

In the unlikely event that the author did not send a complete manuscript
and there are missing pages, these will be noted. Also, if material had to be removed,
a note will indicate the deletion.

UMI®
Dissertation Publishing

UMI 3415373
Copyright 2010 by ProQuest LLC.
All rights reserved. This edition of the work is protected against
unauthorized copying under Title 17, United States Code.

ProQuest®

ProQuest LLC
789 East Eisenhower Parkway
P.O. Box 1346
Ann Arbor, MI 48106-1346

©2010 - Aleksandar Subotic

All rights reserved.

Thesis advisor

Denis Auroux

Author

Aleksandar Subotic

A Monoidal Structure for the Fukaya Category

Abstract

Homological mirror symmetry relates Fukaya categories of certain symplectic spaces to derived categories of coherent sheaves of certain "dual" algebraic spaces. In this work we study the analogue in symplectic geometry of the structure given by the tensor product on the derived category of coherent sheaves, and the symplectic techniques necessary for its definition. We use the theory of triangulated categories with tensor structure to propose that one may think of homological mirror symmetry as the study of how the geometries of certain symplectic manifolds induce natural tensor product structures on their Fukaya categories. This is also the main motivation for this work.

We show that for a Lagrangian torus fibration with a distinguished Lagrangian section, which is the classical setting for mirror symmetry, there exists a natural monoidal structure on a category closely related to the Fukaya category, induced by the group structure on the toric fibers. Having defined the monoidal structure we show that for a mirror pair of elliptic curves, the tensor product can be defined for a version of the Fukaya category used in the proof of homological mirror symmetry for elliptic curves, and that with this tensor structure it will be exactly mirror to the derived category of coherent sheaves of the dual elliptic curve with its standard tensor structure. This gives evidence that our construction agrees with expectations for the mirror tensor product. Also, as a consequence of this theorem one gets a geometric way of computing the ring structure on vector bundles on an elliptic curve.

Contents

Acknowledgments

My biggest thanks go to my advisor Denis Auroux. He is not just an exceptional advisor but also one of the nicest people I know. This short acknowledgement cannot properly describe the full extent of my gratitude and appreciation for him. I am also greatly indebted to my Harvard supervisor Clifford Taubes. He was most generous with his support, and kind enough to take interest in the mathematics I think about while teaching me much along the way. Finally, many thanks to Shing-Tung Yau for all of his support, encouragement and useful discussions throughout my five years at Harvard. He has my special gratitude for being the one to defend my thesis.

There are several other people who helped in the development of this thesis. Special thanks to Paul Balmer, whose suggestions enabled me to find a great motivation for my work. Mohammed Abouzaid, Paul Seidel and Li-Sheng Tseng all had very useful comments that I thank them for. I would also like to thank Denis' other students Sheel Ganatra, Yanki Lekili, Max Maydanskiy and James Pascaleff for many stimulating conversations and seminars.

The support I received was not always mathematical, but it was nevertheless very valuable. Many thanks to Susan Gilbert, Irene Minder and the rest of the Harvard Mathematics Department staff for their great assistance with all things administrative. Finally a great thank you to all my friends: Katy Korner, Ivana Bozic, Ana Caraiani, Sue-Hyun Choi, David Geraghty, Jesse Kass, Tanya Kobilatskya, Ji-Oon Li, Mihai Manea, Djordje Milicevic, Greta Panova, Ryan Reich, Aaron Silberstein, Ethan Street, Ruifang Song, Andrea Tivig, Valentino Tosatti, Jee-Chul Woo, and everyone else who made the last five years so enjoyable for me.

Dedicated to my parents, Sredoje and Vesna Subotic.

Chapter 1

Introduction

Suppose that we are given a mirror dual pair of spaces X and X_{dual}. We will call $D^{symp}(X)$ the triangulated category obtained as the derived category of an A^{∞}-category (the so called "Fukaya category") whose objects are suitable Lagrangian submanifolds of X, and morphisms are Floer complexes with their A^{∞}-structures. We will call $D^{alg}(X_{dual})$ the bounded derived category of suitable coherent sheaves on X_{dual}. Homological mirror symmetry [7] predicts that these two categories are equivalent, but as an object of algebraic geometry D^{alg} is much better understood than D^{symp}. We know that D^{alg} has rich structure and a range of geometric functors. The intuition behind mirror symmetry tells us that there should be corresponding data associated to D^{symp} but the existence of this data is largely conjectural.

The main goal of this thesis is to find the analogue in symplectic geometry of the structure given by the tensor product on the derived category of coherent sheaves, and to study the symplectic techniques necessary for its definition. The motivation for considering the tensor product structure before other structures on D^{alg} is twofold.

The first reason is that the tensor product on D^{alg} is indispensable in many standard constructions of algebraic geometry one might want to study on the symplectic side.

The second, more conceptual reason, is that once any triangulated category K is equipped with a tensor structure it determines a locally ringed space $\mathrm{Spec}(K)$ [4]. Furthermore, if we take K to be $D^{alg}(X_{dual})$ the space $\mathrm{Spec}(K)$ is just X_{dual} [3]. So given the right monoidal structure on $D^{symp}(X)$, one obtains its mirror X_{dual} as $\mathrm{Spec}(D^{symp}(X))$. We propose that one may think of homological mirror symmetry as the study of how the geometries of certain symplectic manifolds X induce natural monoidal structures on $D^{symp}(X)$.

We show that for a Lagrangian torus fibration X with a distinguished Lagrangian section, which is the classical setting for mirror symmetry, there exists a natural monoidal structure on $\mathrm{Don}^{\sharp}(X)$ (a category related to $D^{symp}(X)$) induced by the group structure on the toric fibers. This gives the following theorem

Theorem 1.0.1. *Suppose that X is a Lagrangian torus fibration over a compact connected base B, with a Lagrangian section Σ. Then there is a functor $\hat{\otimes} : \mathrm{Don}^{\sharp}(X) \times \mathrm{Don}^{\sharp}(X) \to \mathrm{Don}^{\sharp}(X)$ that together with the object $(\Sigma, \mathcal{O})$, where $\mathcal{O}$ is the trivial rank 1 local system on $\mathcal{L}$, induces a symmetric monoidal structure on $\mathrm{Don}^{\sharp}(X)$. This functor restricts to fiberwise addition on the level of objects which are Lagrangian sections of the fibration.*

Having defined the monoidal structure we show that for a mirror pair of elliptic curves, the tensor product can be defined for a version of D^{symp} used in the proof of homological mirror symmetry for elliptic curves, and that with this tensor structure D^{symp} will be exactly mirror to D^{alg} with its standard tensor structure.

Theorem 1.0.2. *Let E be a 2-torus with complexified Kähler parameter $\rho = iA + b$, where A is its area. Let E_{dual} be an elliptic curve with modular parameter $\tau = \rho$, and let D^{alg} be the bounded derived category of coherent sheaves on E_{dual} considered as a monoidal category with monoidal structure induced by the standard tensor product. Let D^{symp} be the extended Donaldson category of E equipped with the monoidal structure induced by $\hat{\otimes}$. Then D^{alg} and D^{symp} are equivalent as monoidal categories.*

This gives evidence that our construction agrees with expectations for the mirror tensor product. Also, as a consequence of this theorem one gets a nice geometric way to see the ring structure on vector bundles on an elliptic curve [1].

The main question for future work is how beneficial this setting is to considering fibrations with singular fibers, the much less understood case of mirror symmetry. We end the introduction with a conjecture that expresses this goal:

Conjecture 1.0.3. *If X is SYZ-fibered (admitting a suitable, possibly singular, Lagrangian torus fibration) there is a natural monoidal structure on $D^{symp}(X)$ induced by the geometry of the given SYZ fibration on X.*

Chapter 2

Categories with tensor structure

To study the question of the symplectic mirror of the tensor product of coherent sheaves we need to consider the abstract algebraic structure that extends the tensor product to the level of categories (see for example [9]).

Definition 2.0.4. *A category K has the structure of a* ***monoidal (tensor) category*** *if there exists a bifunctor $\otimes : K \times K \to K$ called the tensor product, and an object I called the unit object so that there are natural isomorphisms:*

1. $\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$
2. $\lambda_A : I \otimes A \cong A$ *and* $\rho_A : A \otimes I \cong A$

that satisfy the following commutative diagrams for any objects A, B, C, D of C:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A\otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C} \otimes \mathrm{Id}"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha_{A,B,C\otimes D}"] & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "\mathrm{Id} \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

and

$$(A \otimes I) \otimes B \xrightarrow{\alpha_{A,I,B}} A \otimes (I \otimes B)$$

$$A \otimes B$$

A monoidal category is particularly simple when all the isomorphisms defining it are identity.

Definition 2.0.5. *A monoidal category is called **strict** if the isomorphisms $\alpha_{A,B,C}, \rho_A, \lambda_A$ are all identity morphisms.*

Example 2.0.6. *The category of modules over a commutative ring R, with the tensor product of modules $\otimes$ is a basic example of a monoidal category. Special cases of this are the categories of vector spaces and abelian groups. We see how naturally appearing monoidal categories are not necessary strict.*

The following example will be of great interest to us.

Example 2.0.7. *Let X be a smooth projective variety, and let $Coh(X)$ be the category of coherent sheaves on X. We will call the bounded derived category of this abelian category $D^b_{coh}(X)$ the **derived category of** X. The category $D^b_{coh}(X)$ is naturally equipped with a tensor product. The tensor product of sheaves is a bi-functor $\otimes : Coh(X) \times Coh(X) \to Coh(X)$ that induces a bi-functor $\otimes^L : D^b_{coh}(X) \times D^b_{coh}(X) \to D^b_{coh}(X)$. The bi-functor $\otimes^L$ and the object $\mathcal{O}_X$ induce a monoidal structure on $D^b_{coh}(X)$.*

The categories we consider will have a symmetric tensor product. More formally

Definition 2.0.8. *A **braided monoidal category** consists of a monoidal category K and a natural isomorphism called the **braiding***

$$B_{A,B} : A \otimes B \to B \otimes A \tag{2.1}$$

so that for any three objects A, B, C in K

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\alpha^{-1}_{A,B,C}"] \arrow[d, "B_{A,B\otimes C}"'] & (A \otimes B) \otimes C \arrow[r, "B_{A,B}\otimes \mathrm{Id}"] & (B \otimes A) \otimes C \arrow[d, "\alpha_{B,A,C}"] \\
(B \otimes C) \otimes A \arrow[r, "\alpha_{B,C,A}"] & B \otimes (C \otimes A) \arrow[r, "\mathrm{Id}\otimes B_{C,A}"] & B \otimes (A \otimes C)
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "B_{A\otimes B,C}"'] & A \otimes (B \otimes C) \arrow[r, "\mathrm{Id}\otimes B_{B,C}"] & A \otimes (C \otimes B) \arrow[d, "\alpha^{-1}_{A,C,B}"] \\
C \otimes (A \otimes B) \arrow[r, "\alpha_{C,A,B}"] & (C \otimes A) \otimes B \arrow[r, "B_{C,A}\otimes \mathrm{Id}"] & (A \otimes C) \otimes B
\end{tikzcd}$$

Definition 2.0.9. *A **symmetric monoidal category** is a braided monoidal category K for which the braiding satisfies $B_{A,B} = B^{-1}_{B,A}$ for all objects A and B in K.*

The functors which respect structure are given as follows

Definition 2.0.10. *A **monoidal functor** from a monoidal category $(K, \otimes, I, \alpha, \lambda, \rho)$ to a monoidal category $(K', \otimes', I', \alpha', \lambda', \rho')$ consists of a functor $F : K \to K'$, a natural isomorphism*

$$\eta_{A,B} : F(A) \otimes' F(B) \to F(A \otimes B) \tag{2.2}$$

for every pair of objects A, B and an isomorphism $\eta_{\mathrm{Id}} : I' \to F(I)$ so that for all objects A, B, C in K the diagrams

$$\begin{tikzcd}
(F(A) \otimes' F(B)) \otimes' F(C) \arrow[r, "\eta_{A,B}\otimes' \mathrm{Id}"] \arrow[d, "\alpha_{F(A),F(B),F(C)}"'] & F(A \otimes B) \otimes' F(C) \arrow[r, "\eta_{A\otimes B,C}"] & F((A \otimes B) \otimes C) \\
F(A) \otimes' (F(B) \otimes' F(C)) \arrow[r, "\mathrm{Id}\otimes' \eta_{B,C}"] & F(A) \otimes' F(B \otimes C) \arrow[r, "\eta_{A,B\otimes C}"] & F(A \otimes (B \otimes C)) \arrow[u, "F(\alpha_{A,B,C})"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
I' \otimes' F(A) \arrow[r, "\lambda'_{F(A)}"] \arrow[d, "\eta_{\mathrm{Id}} \otimes' \mathrm{Id}"'] & F(A) \\
F(I) \otimes' F(A) \arrow[r, "\eta_{I,A}"] & F(I \otimes A) \arrow[u, "F(\lambda_A)"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(A) \otimes' I' \arrow[r, "\rho'_{F(A)}"] \arrow[d, "\mathrm{Id} \otimes' \eta_{\mathrm{Id}}"'] & F(A) \\
F(A) \otimes' F(I) \arrow[r, "\eta_{A,I}"] & F(A \otimes I) \arrow[u, "F(\rho_A)"']
\end{tikzcd}$$

commute.

Definition 2.0.11. *A monoidal functor F is called* ***braided monoidal*** *if it is monoidal and it makes the following diagram commute for all A, B in K*

$$\begin{tikzcd}
F(A) \otimes' F(B) \arrow[r, "B'_{F(A),F(B)}"] \arrow[d, "\eta_{A,B}"'] & F(B) \otimes F(A) \arrow[d, "\eta_{B,A}"'] \\
F(A \otimes B) \arrow[r, "B_{A,B}"] & F(B \otimes A)
\end{tikzcd}$$

Definition 2.0.12. *Suppose F and G are monoidal functors from a monoidal category K to a monoidal category K'. A* ***monoidal natural transformation*** *is a natural transformation $\alpha : F \to G$ so that for any pair of objects A, B in K we have*

$$\begin{tikzcd}
F(A) \otimes' F(B) \arrow[r, "\alpha_A \otimes' \alpha_B"] \arrow[d, "\eta^F_{A,B}"'] & G(A) \otimes' G(B) \\
F(A \otimes B) \arrow[r, "\alpha_{A \otimes B}"] & G(A \otimes B) \arrow[u, "\eta^G_{A,B}"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
I' \arrow[r, "\eta^G_{\mathrm{Id}}"] \arrow[d, "\eta^F_{\mathrm{Id}}"'] & {} \\
F(I) \arrow[r, "\alpha(I)"] & G(I)
\end{tikzcd}$$

commute.

Finally we can state what it means for two monoidal categories to be equivalent.

Definition 2.0.13. *If K and K' are monoidal categories, a monoidal functor $F : K \to K'$ is a* ***monoidal equivalence*** *if there is a monoidal functor $G : K' \to K$ such that there exist monoidal natural isomorphisms between FG and Id_K, and GF and $Id_{K'}$. If such a functor exists we say K and K' are* ***monoidal equivalent.***

Definition 2.0.14. *If K and K' are symmetric monoidal categories, a monoidal equivalence F is symmetric if F and G are symmetric monoidal.*

The importance of monoidal categories to mirror symmetry comes from the following construction.

Definition 2.0.15. *A triangulated category K is* ***tensor triangulated*** *if it admits a symmetric monoidal structure which is exact in each variable.*

In analogy with algebraic geometry one can construct prime ideal subcategories of such a K to obtain a locally ringed space $\mathrm{Spec}(K)$ [4].

Theorem 2.0.16. *[3] Let Y be a topologically noetherian scheme. Let $D^{\mathrm{perf}}(Y)$ be the derived category of perfect complexes over Y. Then*

$$\mathrm{Spec}(D^{\mathrm{perf}}(Y)) \cong Y \tag{2.3}$$

as schemes.

Consider the triangulated category $D^{symp}(X)$ of a symplectic manifold X that admits a mirror algebraic variety Y. In other words suppose there is a nonempty set of varieties Y_i so that the categories $D^{alg}(Y_i) := D^b_{coh}(Y_i)$ are equivalent to $D^{symp}(X)$.

Pulling back the monoidal structures from the $D^{alg}(Y_i)$ for all the Y_i gives a distinguished group of monoidal structures on $D^{symp}(X)$ whose spectra are mirrors to X.

Conjecture 2.0.17. *If X is SYZ-fibered (admitting a suitable, possibly singular, Lagrangian torus fibration) there is a natural monoidal structure on $D^{symp}(X)$ induced by the geometry of the given SYZ fibration on X.*

One can try to go even further. It is an open question as to when $\mathrm{Spec}(K)$ of an arbitrary triangulated monoidal category K admits a scheme structure. If the resulting space is nice one might ask is if there was a natural equivalence from K to $D^{alg}(\mathrm{Spec}(K))$. Answering these algebraic questions would allow one to recognize when an arbitrary symplectic manifold admits a mirror and what the mirror map is.

Chapter 3

The Strominger-Yau-Zaslow transformation in the semi-flat case

In this section we will explain the one known way of geometrically understanding mirror symmetry. In its simplest form the SYZ transform ([13]) is obtained in pairs of special non-singular torus fibrations. We call this the **semi-flat case**, and we largely focus on it without going into further generality. For geometric explanations on how to go beyond the semi-flat case see [2].

Let $M \cong \mathbb{Z}^n$ be a rank n lattice, and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Let D be an open set in $\mathbb{R}^n$. Consider $D \times M_{\mathbb{R}}$ with the natural complex coordinates $x_j + iy_j$ for $j = 1, 2, ..., n$ and x_j and y_j coordinates on D and $M_{\mathbb{R}}$ respectively. There is a natural holomorphic volume form given by $d(x_1 + iy_1) \wedge d(x_2 + iy_2) \wedge ... \wedge d(x_n + iy_n)$.

Call Y the quotient of this space by the action of M. This comes equipped with natural complex coordinates $\exp(x_j + iy_j)$, and holomorphic volume form $\Omega_Y = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge ... \wedge \frac{dz_n}{z_n}$. Furthermore we can equip Y with a Kähler form $\omega_Y = \sum_{j,k} \phi_{j,k} dx_j \wedge dy_k$

for any solution ϕ of

$$\det(\frac{\partial^2\phi}{\partial x_j \partial x_k}) = \text{const} \tag{3.1}$$

This gives Y the structure of a special Lagrangian torus fibration over the base D.

The SYZ mirror X of Y is given by $D \times (N_{\mathbb{R}}/N)$ where N is the dual lattice of M. X is also a special Lagrangian torus fibration over D. One can take ω_X to be $\sum_j dx_j \wedge du_j$ where u_j is the dual coordinate to y_j, with the complex structure given by coordinates w_j determined from $d\log(w_j) = \sum_k \phi_{jk} dx_k + i du_j$ and the holomorphic form $\Omega_X = \frac{dw_1}{w_1} \wedge \frac{dw_2}{w_2} \wedge \ldots \wedge \frac{dw_n}{w_n}$. Notice that in this setup X is naturally symplectic, while Y is naturally complex.

Remark 3.0.18. *A geometric way to think about this is to consider X as the moduli space of pairs (L, ∇), where L is a special Lagrangian torus in M and ∇ is a flat unitary connection on the trivial complex line bundle on L. One can think of the base as representing the Lagrangian part and since the moduli space of flat $U(1)$ connections on the torus is canonically isomorphic to the dual torus we arrive at the dual fibration picture. See [1] for more information.*

The work of Leung-Yau-Zaslow [8] gives some evidence to the fact that this picture induces Homological Mirror Symmetry with the following observation. Let L be a Lagrangian section of the fibration X equipped with a flat $U(1)$ bundle α (thinking of this as one of the basic objects of the Fukaya category of X). We will write it as

$$L = \{(x, u(x)) | x \in D\} \tag{3.2}$$

For every $x \in D$ then $u(x)$ determines a point in $T_N = T_M^{\vee}$ and thus a flat $U(1)$ connection on N. These patch together to give a connection on the topologically trivial complex line bundle over Y given explicitly as

$$\nabla_L = d_Y - \frac{i}{2} \sum_j u_j(x) dy_j \tag{3.3}$$

One can compute that this connection is integrable precisely when L is Lagrangian. Thus, it determines a holomorphic line bundle. We can write down the flat $U(1)$ connection α on L explicitly as $d_L + c$, where c is a closed one-form, and

$$\nabla_{L,\alpha} = \nabla_L + c \tag{3.4}$$

defines another connection integrable if and only if L is Lagrangian. We take the holomorphic line bundle determined by $\nabla_{L,\alpha}$ to be the mirror of (L, α).

What does this tell us about the tensor product? Let L_1 and L_2 be two Lagrangian sections of the fibration X, given by u_1 and u_2 respectively. The hermitian connection determining the tensor product of their mirror holomorphic line bundles is

$$d_Y - \frac{i}{2} \sum_j (u_1(x) + u_2(x)) dy_j \tag{3.5}$$

The suggestion is that on line bundles the geometric tensor product should be given by fiberwise addition.

This is largely the extent of what we know about the SYZ transform. Fortunately there is a natural Lagrangian correspondence from $X \times X$ to X that is determined by the structure of a torus fibration (where tori are considered as groups), and that recovers this behavior.

Chapter 4

Lagrangian correspondences and Wehrheim-Woodward theory

4.0.1 Lagrangian correspondences

Let (M_0, ω_0) and (M_1, ω_1) be two symplectic manifolds. A **Lagrangian correspondence** from M_0 to M_1 is a Lagrangian submanifold of $(M_0 \times M_1, -\omega_0 + \omega_1)$. We will use the notation $M_0^- \times M_1$ to denote $(M_0 \times M_1, -\omega_0 + \omega_1)$.

Example 4.0.19. *Let (M, ω) be a symplectic manifold. Any Lagrangian $L \subset M$ is a Lagrangian correspondence from* pt *to M. The graph of any symplectomorphism $\phi : M \to M$ is a Lagrangian correspondence from M to M.*

In a sense, Lagrangian correspondences from M_0 to M_1 are generalized symplectic morphisms. As such we expect them to induce an action on Lagrangians in M_0:

Definition 4.0.20. *Let L_0 be a Lagrangian in M_0 and L_{01} a Lagrangian correspon-*

dence from M_0 to M_1. The composition $L_0 \circ L_{01}$ is defined to be

$$\pi((L_0 \times L_{01}) \cap (\Delta_{M_0} \times M_1)) \tag{4.1}$$

where $\pi : M_0 \times M_0 \times M_1 \to M_1$ is the projection and Δ_{M_0} is the diagonal in $M_0 \times M_0$.

More generally let (M_2, ω_2) be another symplectic manifold. Let L_{12} be a correspondence from M_1 to M_2. We define $L_{01} \times_{M_1} L_{12}$ to be

$$(L_{01} \times L_{12}) \cap (M_0 \times \Delta_{M_1} \times M_2) \tag{4.2}$$

Then the composition $L_{01} \circ L_{12}$ is defined as

$$\pi(L_{01} \times_{M_1} L_{12}) \tag{4.3}$$

where now $\pi : M_0 \times M_1 \times M_1 \times M_2 \to M_0 \times M_2$

The composition of two Lagrangians is not necessarily a Lagrangian submanifold. The first obstacle is that $L_{01} \times_{M_1} L_{12}$ is not necessarily a transverse intersection. This can be corrected by a Hamiltonian isotopy so since our interest is in hamiltonian isotopy invariant cohomology theories we will assume this is the case. Then

Lemma 4.0.21. *[17] The map $\pi|_{L_{01} \times_{M_1} L_{12}}$ is a Lagrangian immersion.*

When $\pi|_{L_{01} \times_{M_1} L_{12}}$ is an embedding we say that the composition $L_{01} \circ L_{12}$ is **embedded**. To go further than the embedded case we need to extend the definitions.

Definition 4.0.22. *[17] Let M_0 and M_1 be symplectic manifolds. A **generalized Lagrangian correspondence from** M_0 **to** M_1 is a sequence of symplectic manifolds $(N_0, N_1, ..., N_{k-1})$ so that $N_0 = M_0$ and $N_{k-1} = M_1$, and a sequence of Lagrangians*

$(L_{01}, L_{12}, ..., L_{(k-2)(k-1)})$, *so that each* $L_{i(i+1)}$ *is a Lagrangian correspondence from* N_i *to* N_{i+1}. *We denote generalized Lagrangian correspondences by* $\underline{L}$

For a symplectic manifold M *a* ***generalized Lagrangian*** *in* M *is a generalized Lagrangian correspondence from a* pt *to* M.

Our next goal is to define an appropriate Floer homology theory for these generalized Lagrangians. We will need to extend the discussion from [17] in order to include local systems and Novikov coefficients.

4.0.2 Quilted Floer homology with local systems

Consider a generalized Lagrangian correspondence $\underline{L} = (L_{01}, L_{12}, ..., L_{(k-2)(k-1)})$. A **local system on** $\underline{L}$ is a sequence of local systems $\underline{S} := (\mathcal{S}_{01}, \mathcal{S}_{12}, ...\mathcal{S}_{(k-2)(k-1)})$ so that $\mathcal{S}_{i(i+1)}$ is a local system on $L_{i(i+1)}$.

Suppose that we have a generalized Lagrangian correspondence with a local system $(\underline{L}, \underline{S})$. We define the **dual generalized Lagrangian correspondence with a local system** $(\underline{L}, \underline{S})^\vee$ to be $(\underline{L}^\vee, \underline{S}^\vee)$, where $\underline{L}^\vee$ is the reversed Lagrangian correspondence and $\underline{S}^\vee$ is $(\mathcal{S}^\vee_{(k-2)(k-1)}, \mathcal{S}^\vee_{(k-3)(k-2)}, ..., \mathcal{S}^\vee_{01})$. We define the **product of Lagrangian correspondences with local systems** $(L_{01}, \mathcal{S}_{01}) \times (L_{12}, \mathcal{S}_{12})$ to be $(L_{01} \times L_{12}, \mathcal{S}_{01} \boxtimes \mathcal{S}_{12})$, where $\boxtimes$ is just the tensor product of pullbacks of sheaves under the projection maps.

Suppose that we have two Lagrangian correspondences with local systems $(L_{01}, \mathcal{S}_{01})$ and $(L_{12}, \mathcal{S}_{12})$. Suppose that $L_{01} \times_{M_1} L_{12}$ is a transverse intersection locus. Then we equip it with the local system $i^*(\mathcal{S}_{01} \boxtimes \mathcal{S}_{12})$, where $i : L_{01} \times_{M_1} L_{12} \to L_{01} \times L_{12}$ is the inclusion. If $\pi|_{L_{01} \times_{M_1} L_{12}}$ is an embedding we equip $L_{01} \circ L_{12}$ with the local system $\pi_* i^*(\mathcal{S}_{01} \boxtimes \mathcal{S}_{12})$, and we will denote this local system by $\mathcal{S}_{01} \circ \mathcal{S}_{12}$. (Equivalently if one thinks of local systems as representations of the fundamental group, we consider the tensor product representation, then the pullback and then the induced representation.)

Let $(\underline{L}, \underline{S})$ be a cyclic ($N_0 = N_k$) generalized Lagrangian correspondence with local systems. In order our Floer homology to be well defined we will consider manifolds N_i with trivial π_2 and Lagrangian correspondences whose π_1 injects into those of the product symplectic manifolds. Alternatively one can look at [17] where a monotone setup is considered, and where gradings and coefficients are explained in more detail. If $\underline{L} = (L_{01}, L_{12}, ..., L_{(k-1)k})$ has even length we define $HF^*(\underline{L}, \underline{S})$ to be

$$HF^*((L_{(0)}, \mathcal{S}_{(0)})^\vee, (L_{(1)}, \mathcal{S}_{(1)})^T) \tag{4.4}$$

where

$$(L_{(0)}, \mathcal{S}_{(0)}) := (L_{01}, \mathcal{S}_{01}) \times (L_{23}, \mathcal{S}_{23}) \times ... \times (L_{(k-2)(k-1)}, \mathcal{S}_{(k-2)(k-1)}) \tag{4.5}$$

and

$$(L_{(1)}, \mathcal{S}_{(1)}) := (L_{12}, \mathcal{S}_{12}) \times (L_{34}, \mathcal{S}_{34}) \times ... \times (L_{(k-1)k}, \mathcal{S}_{(k-1)k}) \tag{4.6}$$

and ^{T} is induced by the isomorphism $N_1^- \times N_2 \times ... N_{k-1}^- \times N_k \cong N_0 \times N_1^- \times N_2 \times ... N_{k-1}^-$.

If the length is odd we take

$$(L_{(0)}, \mathcal{S}_{(0)}) := (L_{01}, \mathcal{S}_{01}) \times (L_{23}, \mathcal{S}_{23}) \times ... \times (L_{(k-1)k}, \mathcal{S}_{(k-1)k}) \tag{4.7}$$

and

$$(L_{(1)}, \mathcal{S}_{(1)}) := (L_{12}, \mathcal{S}_{12}) \times (L_{34}, \mathcal{S}_{34}) \times ... \times (L_{(k-2)(k-1)}, \mathcal{S}_{(k-2)(k-1)}) \times (\Delta_0, \mathcal{O}) \tag{4.8}$$

where Δ_0 is the diagonal Lagrangian correspondence from N_0 to N_0 with the trivial local system $\mathcal{O}$. We define $HF^*(\underline{L}, \underline{S})$ as before.

Now we define the quilted Floer homology with coefficients in local systems and we will show that the two match. Assume that we have already perturbed the correspondence by an appropriate Hamiltonian action as in [17]. If the length is even we define the quilted complex $QCF^*(\underline{L}, \underline{S})$ to be

$$\bigoplus_{<\underline{x}>} < \underline{x} > \otimes \mathrm{Hom}((\mathcal{S}_{01}|_{(n_0,n_1)})^\vee, \mathcal{S}_{12}|_{(n_1,n_2)}) \otimes \mathrm{Hom}((\mathcal{S}_{23}|_{(n_2,n_3)})^\vee, \mathcal{S}_{34}|_{(n_3,n_4)}) \otimes \tag{4.9}$$

$$... \otimes \mathrm{Hom}((\mathcal{S}_{(k-2)(k-1)}|_{(n_{k-2},n_{k-1})})^\vee, \mathcal{S}_{(k-1)k}|_{(n_{k-1},n_k)}) \otimes \Lambda \tag{4.10}$$

where $\underline{x}$ are tuples $(n_0, n_1, ..., n_k) \in N_0 \times N_1 \times ... \times N_k$ such that for each i we have $(n_i, n_{i+1}) \in L_{i(i+1)}$ and $n_k = n_0$. If the length is odd the above expression becomes

$$\bigoplus_{<\underline{x}>} < \underline{x} > \otimes \mathrm{Hom}((\mathcal{S}_{01}|_{(n_0,n_1)})^\vee, \mathcal{S}_{12}|_{(n_1,n_2)}) \otimes \mathrm{Hom}((\mathcal{S}_{23}|_{(n_2,n_3)})^\vee, \mathcal{S}_{34}|_{(n_3,n_4)}) \otimes \tag{4.11}$$

$$\mathrm{Hom}((\mathcal{S}_{(k-3)(k-2)}|_{(n_{k-3},n_{k-2})})^\vee, \mathcal{S}_{(k-2)(k-1)}|_{(n_{k-2},n_{k-1})}) \otimes \tag{4.12}$$

$$\mathrm{Hom}(((\mathcal{S}_{(k-1)k})|_{(n_{k-1},n_k)})^\vee, \mathcal{O}_{(n_k,n_0)}) \otimes \Lambda \tag{4.13}$$

In both cases Λ is the Novikov parameter.

The Floer coboundary operator without local systems is given by $\partial : QCF^* \to QCF^{*+1}$ in [17] as:

$$\partial < \underline{x}_- > := \sum_{\underline{x}_+} (\sum_{\underline{u} \in \mathcal{M}(\underline{x}_-, \underline{x}_+)} \epsilon(\underline{u})) < \underline{x}_+ > \tag{4.14}$$

where $\mathcal{M}(\underline{x}_-, \underline{x}_+)$ is the moduli of index 1 quilted pseudoholomorphic strips (for a detailed definition refer to [16]) between the two intersection points, with each $\underline{u}$ given by a tuple of maps

$$u_j : \mathbb{R} \times [0, \delta_j] \to N_j \tag{4.15}$$

and $\epsilon : \mathcal{M}(\underline{x}_-, \underline{x}_+) \to \{\pm\}$ are determined by orientations.

Let $s_- := s_{01}^- \otimes s_{12}^- \otimes ... \otimes s_{(k-1)k}^-$ be an element of the local system which appears in the definition of $QCF^*(\underline{L}, \underline{S})$ under its identification (valid in both even and odd cases) with

$$\mathcal{S}_{01}|_{(n_0^-, n_1^-)} \otimes \mathcal{S}_{12}|_{(n_1^-, n_2^-)} \otimes ... \otimes \mathcal{S}_{(k-2)(k-1)}|_{(n_{k-2}^-, n_{k-1}^-)} \otimes \mathcal{S}_{(k-1)k}|_{(n_{k-1}^-, n_k^-)} \tag{4.16}$$

In our case we set

$$\partial(< \underline{x}_- > \otimes s_-) := \sum_{\underline{x}_+, s_+, \underline{u}} \epsilon(\underline{u}) < \underline{x}_+ > \otimes \mathcal{P}_{\underline{u}}(s) \otimes T^{\omega(\underline{u})} \tag{4.17}$$

where T is the Novikov parameter and $\mathcal{P}_{\underline{u}}(s)$ is an element of

$$\mathcal{S}_{01}|_{(n_0^+, n_1^+)} \otimes \mathcal{S}_{12}|_{(n_1^+, n_2^+)} \otimes ... \otimes \mathcal{S}_{(k-2)(k-1)}|_{(n_{k-2}^+, n_{k-1}^+)} \otimes \mathcal{S}_{(k-1)k}|_{(n_{k-1}^+, n_k^+)} \tag{4.18}$$

given by

$$\mathcal{P}_{\underline{u}}(s) = (\gamma_{12} \circ (s_{01}^- \otimes s_{12}^-) \circ (\gamma_{01})^\vee) \otimes (\gamma_{34} \circ (s_{23}^- \otimes s_{34}^-) \circ (\gamma_{23})^\vee) \otimes ... \tag{4.19}$$

where each $\gamma_{j(j+1)}$ is the monodromy morphism in

$$\mathrm{Hom}(\mathcal{S}_{j(j+1)}|_{(n_j^-, n_{j+1}^-)}, \mathcal{S}_{j(j+1)}|_{(n_j^+, n_{j+1}^+)}) \tag{4.20}$$

induced by the curve $(u_j(s, \delta_j), u_{j+1}(s, 0))$, and $\gamma_{j(j+1)}^\vee$ is its dual in

$$\mathrm{Hom}((\mathcal{S}_{j(j+1)}|_{(n_j^+, n_{j+1}^+)})^\vee, (\mathcal{S}_{j(j+1)}|_{(n_j^-, n_{j+1}^-)})^\vee) \tag{4.21}$$

Notice that in the odd case we consider the quilt as having an "extra seam condition" given by the pair $(\Delta_0, \mathcal{O})$. This observation is really a part of the following proposition

Proposition 4.0.23. *Quilted Floer homology with local coefficients is well defined and agrees with generalized Floer homology with local coefficients.*

Proof. That the quilted Floer homology with local coefficients is well defined follows from an argument similar to that in [17]. The second part of the claim will follow from unraveling the definitions.

As is stated in [17] the strips contributing to the differential in $HF(\underline{L}, \underline{S})$ are in a 1-1 correspondence with the quilts of $QHF(\underline{L}, \underline{S})$. Consider such a J-holomorphic strip $w(s,t)$ with $s \to -\infty$ limit at the intersection point $(n_0^-, ..., n_{k-1}^-)$ and the $s \to +\infty$ limit at $(n_0^+, ..., n_{k-1}^+)$. Unfolding this curve gives a quilt $\underline{u}$ by taking

$$w(s,t) = (u_0(s, 1-t), u_1(s,t), u_2(s, 1-t), ...) \tag{4.22}$$

where all the

$$(u_j(s,1), u_{j+1}(s,0)) \in L_{j(j+1)} \tag{4.23}$$

and $u_k := u_0$.

Let t an element of

$$\mathrm{Hom}((\mathcal{S}_{01}^\vee)|_{(n_0^-, n_1^-)} \otimes (\mathcal{S}_{23}^\vee)|_{(n_2^-, n_3^-)} \otimes ..., \mathcal{S}_{12}|_{(n_1^-, n_2^-)} \otimes \mathcal{S}_{34}|_{(n_3^-, n_4^-)} \otimes ...) \tag{4.24}$$

Compared to the case with no local systems now every strip w contributes an extra $\mathcal{P}_w(t)T^{\omega(w)}$ where

$$\mathcal{P}_w(t) = \gamma_1 \circ t \circ \gamma_0^\vee \tag{4.25}$$

with $\gamma_0^\vee$ being the dual of the monodromy morphism $\gamma_0^\vee$ along $w(s,0)$ inducing a morphism in

$$\mathrm{Hom}(\mathcal{S}_{01}|_{(n_0^-, n_1^-)} \otimes \mathcal{S}_{23}|_{(n_2^-, n_3^-)} \otimes ..., \mathcal{S}_{01}|_{(n_0^+, n_1^+)} \otimes \mathcal{S}_{23}|_{(n_2^+, n_3^+)} \otimes ...) \tag{4.26}$$

and γ_1 being the monodromy morphism along $w(s,1)$ inducing a morphism in

$$\mathrm{Hom}(\mathcal{S}_{12}|_{(n_1^-, n_2^-)} \otimes \mathcal{S}_{34}|_{(n_3^-, n_4^-)} \otimes ..., \mathcal{S}_{12}|_{(n_1^+, n_2^+)} \otimes \mathcal{S}_{34}|_{(n_3^+, n_4^+)} \otimes ...) \tag{4.27}$$

Under the natural isomorphism of equations [4.16] and [4.24], t can be rewritten as

$$t = s_{01}^{-} \otimes s_{12}^{-} \otimes \ldots \otimes s_{(k-1)k}^{-} \tag{4.28}$$

Similarly, under a natural isomorphism γ_0 can be rewritten as $\gamma_{01} \otimes \gamma_{23} \otimes \ldots$ and γ_1 as $\gamma_{12} \otimes \gamma_{34} \otimes \ldots$ where each $\gamma_{j(j+1)}$ is in $\mathrm{Hom}(\mathcal{S}_{j(j+1)}|_{(n_j^-, n_{j+1}^-)}, \mathcal{S}_{j(j+1)}|_{(n_j^+, n_{j+1}^+)})$

We claim that $\mathcal{P}_w(t) = \mathcal{P}_{\underline{u}}(s)$, that is that

$$(\gamma_{12} \otimes \gamma_{34} \otimes \ldots) \circ (s_{01}^{-} \otimes s_{12}^{-} \otimes \ldots \otimes s_{(k-1)k}^{-}) \circ (\gamma_{01}^{\vee} \otimes \gamma_{23}^{\vee} \otimes \ldots) \tag{4.29}$$

But this is straightforward because

$$w(s,0) = (u_0(s,1), u_1(s,0)) \times (u_2(s,1), u_3(s,0)) \times \ldots \tag{4.30}$$

and so the monodromy morphisms $\gamma_{j(j+1)}$ can be identified precisely with the monodromies along $(u_j(s,1), u_{j+1}(s,0))$. □

Next we prove an extension of the main theorem of [14]

Lemma 4.0.24. *Let $(L_1, \mathcal{S}_1)$, $(L_{12}, \mathcal{S}_{12})$, $(L_{23}, \mathcal{S}_{23})$, $(L_3, \mathcal{S}_3)$ be a sequence of Lagrangian correspondences from (pt, N_1, N_2, N_3, pt). Suppose that the composition $L_{12} \circ L_{23}$ is embedded. Then*

$$HF^*(((L_1, \mathcal{S}_1) \times (L_{23}, \mathcal{S}_{23}))^{\vee}, (L_{12}, \mathcal{S}_{12}) \times (L_3, \mathcal{S}_3)) \cong HF^*(((L_1, \mathcal{S}_1) \times (L_3, \mathcal{S}_3))^{\vee}, (L_{12} \circ L_{23}, \mathcal{S}_{12} \circ \mathcal{S}_{23}) \tag{4.31}$$

Proof. On the level of complexes we have that $CF^*(((L_1, \mathcal{S}_1) \times (L_{23}, \mathcal{S}_{23}))^{\vee}, (L_{12}, \mathcal{S}_{12}) \times (L_3, \mathcal{S}_3))$ equals

$$\oplus_x < x > \otimes \mathrm{Hom}((\mathcal{S}_1 \boxtimes \mathcal{S}_{23})^{\vee}|_x, (\mathcal{S}_{12} \boxtimes \mathcal{S}_3)|_x) \otimes \Lambda \tag{4.32}$$

Let $x = (n_1, n_2, n_3)$. We can decompose $\mathrm{Hom}((\mathcal{S}_1 \boxtimes \mathcal{S}_{23})^\vee|_x, (\mathcal{S}_{12} \boxtimes \mathcal{S}_3)|_x)$ as

$$\mathcal{S}_1|_{n_1} \otimes \mathcal{S}_{23}|_{(n_2,n_3)} \otimes \mathcal{S}_{12}|_{(n_1,n_2)} \otimes \mathcal{S}_3|_{n_3} \tag{4.33}$$

On the other hand $CF^*(((L_1, \mathcal{S}_1) \times (L_3, \mathcal{S}_3))^\vee, (L_{12} \circ L_{23}, \mathcal{S}_{12} \circ \mathcal{S}_{23}))$ equals

$$\oplus_y < y > \otimes \mathrm{Hom}((\mathcal{S}_1 \boxtimes \mathcal{S}_3)^\vee|_y, (\mathcal{S}_{12} \circ \mathcal{S}_{23})|_y) \otimes \Lambda \tag{4.34}$$

Let $y = (m_1, m_3)$. Then we can decompose $\mathrm{Hom}((\mathcal{S}_1 \boxtimes \mathcal{S}_3)^\vee|_y, (\mathcal{S}_{12} \circ \mathcal{S}_{23})|_y)$ as

$$\mathcal{S}_1|_{m_1} \otimes \mathcal{S}_3|_{m_3} \otimes (\mathcal{S}_{12} \circ \mathcal{S}_{23})|_(m_1, m_3) \tag{4.35}$$

Since $L_{12} \circ L_{23}$ is embedded, for every y we can find a unique m_2 so that $(m_1, m_2, m_3) \in (L_1 \times L_{23}) \cap (L_{12} \times L_3)$. Then since $\mathcal{S}_{12} \circ \mathcal{S}_{23} = \pi_* i^*(\mathcal{S}_{12} \boxtimes \mathcal{S}_{23})$ we again get

$$\mathcal{S}_1|_{m_1} \otimes \mathcal{S}_3|_{m_3} \otimes \mathcal{S}_{12}|_{(m_1,m_2)} \otimes \mathcal{S}_{23}|_{(m_2,m_3)} \tag{4.36}$$

On the level of the differentials we use the quilted picture. Let $\underline{u}$ be a quilt contributing to the right hand side and $\underline{v} := \mathcal{T}_\delta(\underline{u})$ where $\mathcal{T}_\delta$ is the isomorphism of the appropriate moduli of quilts constructed in section 3.1 of [14]. To extend the arguments found there it is enough to show that

$$\mathcal{P}_{\underline{u}}(s) = (\gamma_{12} \circ (s_1^- \otimes s_{12}^-) \circ (\gamma_1)^\vee) \otimes (\gamma_3 \circ (s_{23}^- \otimes s_3^-) \circ (\gamma_{23})^\vee) \tag{4.37}$$

equals

$$\mathcal{P}_{\underline{v}}(t) = (\rho_3 \circ (t_1^- \otimes t_3^-) \circ (\rho_1)^\vee) \otimes (t_{123}^- \circ (\rho_{123})^\vee) \tag{4.38}$$

where s and t are in spaces given by [4.35] and [4.37] and γ_j, ρ_j and $\gamma_{j(j+1)}, \rho_{j(j+1)}$ are appropriate holonomies as before.

By the first part s and t can be identified by an isomorphism so that $s_1^- \cong t_1^-$, $s_3^- \cong t_3^-$ and $s_{12}^- \otimes s_{23}^- \cong t_{123}^-$. By construction of $\mathcal{T}_\delta$ there is a homotopy between $\underline{u}$

and $\underline{v}$. More precisely, following the proof of the main theorem of [14] we can identify $\underline{u}(s,t) = (u_1(s,t), u_3(s,t))$ with the tuple $(u_{13}, \overline{u})$ where

$$u_{13}(s,t) = (u_1(s, 1-t), u_3(s,t)) \tag{4.39}$$

is of width 1 and

$$\overline{u}(s,t) = (\overline{u}_{13}(s), \overline{u}_2(s), \overline{u}_2(s)) \tag{4.40}$$

is of width δ with $\overline{u}_{13}(s) = (u_1(s,1), u_3(s,0))$, $\overline{u}_2 = l_2 \circ \overline{u}_{13}$, where l_2 is the M_2 component of the inverse of $\pi : L_{12} \times_{M_2} L_{23} \to L_{12} \circ L_{23}$.

We can identify $\underline{v}(s,t) = (v_1(s,t), v_2(s,t), v_3(s,t))$ with the tuple

$$(v_{13}(s,t), v_{13}'(s,t), v_2(s,t), v_2'(s,t)) \tag{4.41}$$

where

$$(v_1(s, 1-t), v_3(s,t)) = \begin{cases} v_{13}'((1+\bar{\delta})s, \bar{\delta} - (1+\bar{\delta})t), & \text{for } 0 \le t \le \frac{1}{2}\delta; \\ v_{13}((1+\bar{\delta})s, (1+\bar{\delta})t - \bar{\delta}), & \text{for } \frac{1}{2}\delta \le t \le 1. \end{cases} \tag{4.42}$$

and

$$v_2(s,t) = \begin{cases} v_2'((1+\bar{\delta})s, \bar{\delta} - (1+\bar{\delta})t), & \text{for } 0 \le t \le \frac{1}{2}\delta; \\ v_2((1+\bar{\delta})s, (1+\bar{\delta})t - \bar{\delta}), & \text{for } \frac{1}{2}\delta \le t \le \delta. \end{cases} \tag{4.43}$$

The number δ is the width of the middle strip in $\underline{v}$ (the other ones are width 1) and $\bar{\delta} := \frac{\delta}{2-\delta}$. It is for these tuples of strips that there is a homotopy. We can immediately see that the curves $v_{13}(s,1) = (v_1(\frac{1}{1+\bar{\delta}}s, 0), \frac{1}{1+\bar{\delta}}v_3(s,1))$ and $u_{13}(s,1) = (u_1(s,0), u_3(s,1))$ are homotopic so we can identify the ρ_j's and γ_j's.

For the rest, ρ_{123} induced by $\overline{u}_{13}$ is the same as the holonomy of the local system $\mathcal{S}_{12} \boxtimes \mathcal{S}_{23}$ along $\overline{u}$. On the other hand γ_{12}, γ_{23} are induced by $(v_1(s,1), v_2(s,0))$

and $(v_2(s,\delta), v_3(s,0))$ respectively. In terms of the new tuples the last two become $(v_{13}'((1+\bar{\delta})s,\bar{\delta}), v_2'((1+\bar{\delta})s,\bar{\delta}), v_2((1+\bar{\delta})s,\bar{\delta}))$. This curve induces a holonomy of $\mathcal{S}_{12} \boxtimes \mathcal{S}_{23}$ that is by homotopy the same as the one given by $\overline{u}$. But the latter one is also equal to the one given by $\gamma_{23} \circ (s_{12}^{-} \otimes s_{23}^{-}) \circ (\gamma_{12})^{\vee}$ after using the isomorphism of

$$\mathrm{Hom}(\mathcal{S}_{12}|_{(m_1^-,m_2^-)} \otimes \mathcal{S}_{23}|_{(m_2^-,m_3^-)}, \mathcal{S}_{12}|_{(m_1^+,m_2^+)} \otimes \mathcal{S}_{23}|_{(m_2^+,m_3^+)}) \tag{4.44}$$

and

$$\mathrm{Hom}((\mathcal{S}_{12}|_{(m_1^+,m_2^+)})^{\vee}, (\mathcal{S}_{12}|_{(m_1^-,m_2^-)})^{\vee}) \otimes \mathrm{Hom}((\mathcal{S}_{23}|_{(m_2^-,m_3^-)}), (\mathcal{S}_{23}|_{(m_2^+,m_3^+)})) \tag{4.45}$$

Thus putting it all together

$$t_{123} \circ (\rho_{123})^{\vee} = \gamma_{23} \circ (s_{12} \otimes s_{23}) \circ (\gamma_{12})^{\vee} \tag{4.46}$$

□

4.0.3 Donaldson-Fukaya categories with local systems

Next we review and extend the definition of the Donaldson-Fukaya categories of generalized Lagrangians.

We will need the following theorem, which is a straightforward extension of arguments in [16] using ideas of the last chapter

Theorem 4.0.25. *Given a* ***quilted surface with strip-like ends*** $\underline{C}$*, and a collection of Lagrangian submanifolds with local systems* $(\underline{L}_e, \underline{S}_e)$ *for the seams and boundaries of* $\underline{C}$*, one for each end* $\underline{e}$*, we can define a* ***relative invariant*** $\Phi_{\underline{C}}$ *that counts the number of pseudoholomorphic quilts with these Lagrangian boundary conditions.*

and gives a map

$$\Phi_{\underline{C}} : \otimes_{\underline{e}^-} HF((\underline{L}_{\underline{e}^-}, \underline{S}_{\underline{e}^-})) \to \otimes_{\underline{e}^+} HF((\underline{L}_{\underline{e}^+}, \underline{S}_{\underline{e}^+})) \tag{4.47}$$

Furthermore, if $\underline{C}'$ is a quilted surface obtained by removing a patch between Lagrangian correspondences whose composition is embedded and replacing it with a new seam corresponding to their composition, then the isomorphisms

$$\Psi_{\underline{e}} : HF((\underline{L}_{\underline{e}}, \underline{S}_{\underline{e}})) \to HF((\underline{L}'_{\underline{e}}, \underline{S}'_{\underline{e}})), \tag{4.48}$$

where $(\underline{L}'_{\underline{e}}, \underline{S}'_{\underline{e}})$ are the new boundary conditions at each end, intertwine with the relative invariants in the sense that

$$\Phi_{\underline{C}'} \circ (\otimes_{\underline{e}^-} \Psi_{\underline{e}^-}) = (\otimes_{\underline{e}^+} \Psi_{\underline{e}^+}) \circ \Phi_{\underline{C}} \tag{4.49}$$

Remark 4.0.26. *Again we refer the reader to [16] for a detailed definition of quilted surfaces with strip like ends and the associated invariant.*

Definition 4.0.27. *The* ***generalized Donaldson-Fukaya category*** $\mathrm{Don}^\sharp(M)$ *of a symplectic manifold M is defined as follows:*

- *The objects of $\mathrm{Don}^\sharp(M)$ are generalized Lagrangians satisfying suitable technical conditions (like the ones discussed in section 4.1) with local systems $(\underline{L}, \underline{S})$ in M.*
- *The morphism space between two pairs $(\underline{L}_1, \underline{S}_1)$ and $(\underline{L}_2, \underline{S}_2)$is*

$$\mathrm{Hom}((\underline{L}_1, \underline{S}_1), (\underline{L}_2, \underline{S}_2)) := HF(\underline{L}, \underline{S})[d] \tag{4.50}$$

where $(\underline{L}, \underline{S})$ is the periodic correspondence given by concatenating $(\underline{L}_1, \underline{S}_1)$ and $(\underline{L}_2, \underline{S}_2)^\vee$, and $d = \frac{1}{2}(\sum_k \dim(N_k) + \sum_{k'} \dim(N'_{k'}))$.

- *The composition of morphisms in* $\mathrm{Don}^\sharp(M)$ *is given by the relative invariant* $\Phi_{\underline{P}}$ *corresponding to the quilted half pair of pants* $\underline{P}$

$$\mathrm{Hom}((\underline{L}_1, \underline{S}_1), (\underline{L}_2, \underline{S}_2)) \times \mathrm{Hom}((\underline{L}_2, \underline{S}_2), (\underline{L}_3, \underline{S}_3)) \to \mathrm{Hom}((\underline{L}_1, \underline{S}_1), (\underline{L}_3, \underline{S}_3)) \tag{4.51}$$

- *The identities* $1_{(\underline{L},\underline{S})} \in \mathrm{End}((\underline{L}, \underline{S}))$ *are given by relative invariants associated to the quilted disk [16].*

As a consequence of the previous theorem we obtain the following lemma

Lemma 4.0.28. *Let* $(L_{01}, L_{12}, ..., L_{(k-2)(k-1)})$ *be a generalized Lagrangian with local systems (suppressing the local systems from the expanded notation) such that* $L_{(j-1)j} \circ L_{j(j+1)}$ *is embedded. Then it is equivalent to the generalized Lagrangian with local systems* $(L_{01}, L_{12}, ..., L_{(j-1)j} \circ L_{j(j+1)}, ..., L_{(k-2)(k-1)})$ *as an object of* $\mathrm{Don}^\sharp(M)$.

Furthermore, let M_0 and M_1 be two symplectic manifolds. A Lagrangian correspondence with local systems $(L_{01}, \mathcal{S}_{01})$ induces a functor $\Phi((L_{01}, \mathcal{S}_{01}))$ from $\mathrm{Don}^\sharp(M_0)$ to $\mathrm{Don}^\sharp(M_1)$, which is well defined and constructed in analogy with [15] and the previous discussion:

Definition 4.0.29. *The functor* $\Phi((L_{01}, \mathcal{S}_{01})) : \mathrm{Don}^\sharp(M_0) \to \mathrm{Don}^\sharp(M_1)$ *is defined by*

- *On the level of objects, taking concatenation of the given object of* $\mathrm{Don}^\sharp(M_0)$ *with the Lagrangian correspondence* $(L_{01}, \mathcal{S}_{01})$

- *On the level of morphisms taking* $\Phi((L_{01}, \mathcal{S}_{01})) := \Phi_{\underline{C}}$*, where* $\underline{C}$ *is a quilted surface with two punctures and one interior circle [15]*

We will call the full subcategory of $\mathrm{Don}^\sharp(M)$ given by objects which are just Lagrangian submanifolds $\mathrm{Don}(M)$. We will need one more functor

Theorem 4.0.30. *There is a fully faithful functor* $i : \mathrm{Don}^\sharp(M) \times \mathrm{Don}^\sharp(M) \to \mathrm{Don}^\sharp(M \times M)$ *that extends the product functor on the subcategory* $\mathrm{Don}(M)$.

Proof. Let $(\underline{L}, \underline{S}) = (L_{01}, ..., L_{(k-2)(k-1)})$ and $(\underline{G}, \underline{T}) = (G_{01}, ..., G_{(m-2)(m-1)})$ be two generalized Lagrangians with local systems (suppressing the local systems from the expanded notation) such that $m < k$. On objects

$$i((\underline{L}, \underline{S}) \times (\underline{G}, \underline{T})) = (L_{01}, L_{12}, ..., L_{(k-m)(k-m+1)} \times G_{01}, ..., L_{(k-2)(k-1)} \times G_{(m-2)(m-1)}) \tag{4.52}$$

Let $(\underline{N}, \underline{P}) = (N_{01}, ..., N_{(j-2)(j-1)})$ and $(\underline{R}, \underline{Q}) = (R_{01}, ..., R_{(s-2)(s-1)})$, with $s < j$ be two other generalized Lagrangians. For morphisms we need to consider

$$\mathrm{Hom}_{\mathrm{Don}^\sharp(M)}((\underline{L}, \underline{S}), (\underline{N}, \underline{P})) \times \mathrm{Hom}_{\mathrm{Don}^\sharp(M)}((\underline{G}, \underline{T}), (\underline{R}, \underline{Q})) \tag{4.53}$$

The first Hom equals (up to a shift)

$$HF^*(A_{(0)}^\vee, A_{(1)}) \tag{4.54}$$

where

$$A_{(0)} = L_{01} \times L_{23} \times ... \times N^T_{\alpha(\alpha+1)} \times N^T_{(\alpha-1)(\alpha-2)} \times ... \tag{4.55}$$

and

$$A_{(1)} = L_{12} \times L_{34} \times ... \times N^T_{\alpha'(\alpha'+1)} \times N^T_{(\alpha'-1)(\alpha'-2)} \times ... \tag{4.56}$$

where $|\alpha' - \alpha| = 1$ are numbers that depend on the parity of k.

Similarly the second Hom equals

$$HF^*(B_{(0)}^\vee, B_{(1)}) \tag{4.57}$$

where

$$B_{(0)} = G_{01} \times G_{23} \times ... \times R^T_{\beta(\beta+1)} \times R^T_{(\beta-1)(\beta-2)} \times ... \tag{4.58}$$

and

$$B_{(1)} = G_{12} \times G_{34} \times ... \times R^T_{\beta'(\beta'+1)} \times R^T_{(\beta'-1)(\beta'-2)} \times ... \tag{4.59}$$

where $|\beta' - \beta| = 1$ are numbers that depend on the parity of m.

By the decomposition property of Floer homology

$$HF^*(A^\vee_{(0)}, A_{(1)}) \otimes HF^*(B^\vee_{(0)}, B_{(1)}) \tag{4.60}$$

equals $HF^*(L^\vee_{(0)}, L_{(1)})$ where $L_{(0)} = A_{(0)} \times B_{(0)}$ and $L_{(1)} = A_{(1)} \times B_{(1)}$.

Let $(\underline{L}, \underline{S})$ and $(\underline{G}, \underline{T})$ be of even length. Then after a rearrangement of factors we get

$$L_{(0)} = L_{01} \times ... \times (L_{(k-2)(k-1)} \times G_{(m-2)(m-1)}) \times (N^T_{(j-3)(j-2)} \times R^T_{(s-3)(s-2)}) \times ... \tag{4.61}$$

and

$$L_{(1)} = L_{12} \times ... \times (L_{(k-3)(k-2)} \times G_{(m-3)(m-2)}) \times (N^T_{(j-2)(j-1)} \times R^T_{(s-2)(s-1)}) \times ... \tag{4.62}$$

Thus we get exactly the Hom space

$$HF^*(i((\underline{L}, \underline{S}) \times (\underline{G}, \underline{T})), i((\underline{N}, \underline{P}) \times (\underline{R}, \underline{Q}))). \tag{4.63}$$

This identification defines our functor on morphisms and at the same time proves that it is fully faithful.

If $(\underline{L}, \underline{S})$ and $(\underline{G}, \underline{T})$ are both of odd length we get the same situation. Let $(\underline{L}, \underline{S})$ be of even length and $(\underline{G}, \underline{T})$ of odd length. Then we get

$$L_{(0)} = L_{01} \times ... \times (L_{(k-2)(k-1)} \times G_{(m-3)(m-2)}) \times (N^T_{(j-3)(j-2)} \times R^T_{(s-2)(s-1)}) \times ... \tag{4.64}$$

and

$$L_{(1)} = L_{12} \times \ldots \times (L_{(k-3)(k-2)} \times G_{(m-2)(m-1)}) \times (N^T_{(j-2)(j-1)} \times R^T_{(s-3)(s-2)}) \times \ldots \quad (4.65)$$

Equivalently this is the Floer homology of the sequence

$$(L_{01}, L_{12}, \ldots, L_{(k-3)(k-2)} \times G_{(m-2)(m-1)}, L_{(k-2)(k-1)} \times R^T_{(s-2)(s-1)}, N^T_{(j-2)(j-1)} \times R^T_{(s-3)(s-2)}, \ldots) \quad (4.66)$$

Lemma 4.0.31. *The Floer homology of the sequence*

$$(L_{01}, L_{12}, \ldots, L_{(k-3)(k-2)} \times G_{(m-2)(m-1)}, L_{(k-2)(k-1)} \times R^T_{(s-2)(s-1)}, N^T_{(j-2)(j-1)} \times R^T_{(s-3)(s-2)}, \ldots) \quad (4.67)$$

can be expressed as the Floer homology of the pair of generalized Lagrangians

$$(L_{01}, L_{12}, \ldots, L_{(k-3)(k-2)} \times G_{(m-2)(m-1)}, L_{(k-2)(k-1)} \times \Delta_M) \quad (4.68)$$

and

$$(\ldots, N_{(j-2)(j-1)} \times R_{(s-3)(s-2)}, \Delta_M \times R_{(s-2)(s-1)}) \quad (4.69)$$

Proof. The claim is equivalent to the fact that we can replace $L_{(k-2)(k-1)} \times R^T_{(s-2)(s-1)}$ with the sequence $(L_{(k-2)(k-1)} \times \Delta_M, \Delta_M \times R^T_{(s-2)(s-1)})$. But this sequence gives an embedded composition and composes exactly to $L_{(k-2)(k-1)} \times R^T_{(s-2)(s-1)}$. □

Lemma 4.0.32. *The two generalized Lagrangians*

$$(L_{01}, L_{12}, \ldots, L_{(k-3)(k-2)} \times G_{(m-2)(m-1)}, L_{(k-2)(k-1)} \times \Delta_M) \quad (4.70)$$

and $i((\underline{L}, \underline{S}), (\underline{G}, \underline{T}))$ *are equivalent as objects of* $\mathrm{Don}^\sharp(M \times M)$.

Proof. Let $(\underline{V}, \underline{W})$ be a test object. The generalized intersection points are obviously in correspondence.

Let $\underline{u}$ be a quilted strip with the seam conditions $(L_{01}, L_{12}, ..., L_{(k-3)(k-2)} \times G_{(m-2)(m-1)}, L_{(k-2)(k}$ $\Delta_M, \underline{V}^T)$. We can write $\underline{u} = (u_1, u_2, ..., u_{k-2} \times v_{k-1}, u_{k-1} \times v_k, ...)$ so that $u_1(s,0) \in L_{01}$, $(u_1(s,1), u_2(s,0)) \in L_{12}$, ..., $(u_{k-3}(s,1), u_{k-2}(s,0)) \in L_{(k-3)(k-2)}$, $(v_{k-2}(s,1), v_{k-1}(s,0)) \in G_{(m-2)(m-1)}$ $(u_{k-2}(s,1), u_{k-1}(s,0)) \in L_{(k-2)(k-1)}$, $(v_{k-1}(s,1), v_k(s,0)) \in \Delta_M$. Define $v' : \mathbb{R} \times [0,1] \to M$ to be the strip given by concatenating $v_{k-1}(s, \frac{1}{2}t)$ and $v_k(s, \frac{1}{2}t)$. Then

$$\underline{u}' = (u_1, u_2, ..., u_{k-2} \times v_{k-2}, u_{k-1} \times v', ...) \tag{4.71}$$

gives a $J_{\frac{1}{2}}$-holomorphic map a quilted strip with seam conditions in $i((\underline{L}, \underline{S}), (\underline{G}, \underline{T}))$, where $J_{\frac{1}{2}}$ is a deformed almost complex structure as per the proof of Theorem 5.2.3 of [17]. From this point on one can follow their argument that shows these are in a $1-1$ correspondence with width 1 quilted strips for a genuine almost complex structure. Similarly, one can argue for quilted triangles which finishes the proof. □

Applying the above lemma to

$$(..., N_{(j-2)(j-1)} \times R_{(s-3)(s-2)}, \Delta_M \times R_{(s-2)(s-1)}) \tag{4.72}$$

as well again establishes that the result is

$$HF^*(i((\underline{L}, \underline{S}) \times (\underline{G}, \underline{T})), i((\underline{N}, \underline{P}) \times (\underline{R}, \underline{Q}))) \tag{4.73}$$

Since there was no loss of generality in choosing $(\underline{L}, \underline{S})$ to be of even length and $(\underline{G}, \underline{T})$ to be of odd length, this finishes the proof. □

Chapter 5

Definition of the tensor product

We are now ready to define the bifunctor giving the tensor product. Consider the set $\mathcal{L}$ in $(X \times X)^{-} \times X$, where X is semi-flat, given by

$$\mathcal{L} = \{(x, u_1, x, u_2, x, u_1 + u_2) | x \in D, \quad u_1, u_2 \in T_M\} \tag{5.1}$$

where $X \cong D \times T_M$.

Lemma 5.0.33. *$\mathcal{L}$ is a smooth Lagrangian correspondence.*

Proof. Under a diffeomorphism that permutes the factors of this product $\mathcal{L}$ is given by $\Delta_{D\times D\times D} \times \Gamma_+$, where Γ_+ is the graph of addition on T_M.

To check that it is Lagrangian

$$\omega_{(X\times X)^{-}\times X}((P, U_1), (P, U_2), (P, U_1 + U_2), (Q, V_1), (Q, V_2), (Q, V_1 + V_2)) = \tag{5.2}$$

$$-\omega_X((P, U_1), (Q, V_1)) - \omega_X((P, U_2), (Q, V_2)) + \omega_X((P, U_1 + U_2), (Q, V_1 + V_2)) \tag{5.3}$$

The simple form $\sum_j dx_j \wedge du_j$ of ω_X for a semi-flat X implies that this is 0.

□

The way this composition acts on pairs of Lagrangians intuitively is by fiberwise Minkowski addition. In other words for every fiber of the fibration on X that both of the Lagrangians intersect, we Minkowski sum the intersection points of one with the intersection points of the other. In particular this recovers the behavior predicted by the SYZ-picture.

We have so far focused on the local picture, which is believed to be a part of the more general mirror symmetry picture. Suppose now that X is a Lagrangian torus fibration with no singular fibers over a compact connected base B, with a Lagrangian section. Then B is affine and X is isomorphic to T^*B/Λ (see [5]) so we can define the Lagrangian correspondence $\mathcal{L}$.

Theorem 5.0.34. *Suppose that X is a Lagrangian torus fibration over a compact connected base B, with a Lagrangian section Σ. Then there is a functor $\hat{\otimes} : \mathrm{Don}^\sharp(X) \times \mathrm{Don}^\sharp(X) \to \mathrm{Don}^\sharp(X)$ that together with the object $(\Sigma, \mathcal{O})$, where Σ is the zero section and $\mathcal{O}$ is the trivial rank 1 local system on $\mathcal{L}$, induces a symmetric monoidal structure on $\mathrm{Don}^\sharp(X)$. This functor restricts to fiberwise addition on the level of objects which are Lagrangian sections of the fibration.*

Proof. Using Duistermaat's theorem we can define the Lagrangian correspondence $\mathcal{L}$ in $X^- \times X^- \times X$. This then defines a functor $\Phi((\mathcal{L}, \mathcal{O}))$ from $\mathrm{Don}^\sharp(X \times X)$ to $\mathrm{Don}^\sharp(X)$. We define

$$\hat{\otimes} := \Phi((\mathcal{L}, \mathcal{O})) \circ i \tag{5.4}$$

and this functor, as discussed, has the desired property of being fiberwise addition on the level of objects.

Let $(\underline{L},\underline{S}) = (L_{01},...,L_{(k-2)(k-1)})$, $(\underline{G},\underline{T}) = (G_{01},...,G_{(m-2)(m-1)})$ and $(\underline{N},\underline{P}) = (N_{01},...,N_{(j-2)(j-1)})$ be 3 objects of $\mathrm{Don}^\sharp(X)$. Then

$$((\underline{L},\underline{S})\hat{\otimes}(\underline{G},\underline{T}))\hat{\otimes}(\underline{N},\underline{P}) \tag{5.5}$$

is given by the sequence

$$(...,L_{(k-2)(k-1)} \times G_{(m-2)(m-1)} \times N_{(j-3)(j-2)}, \mathcal{L} \times N_{(j-2)(j-1)}, \mathcal{L}) \tag{5.6}$$

which by Lemma 4.14 is equivalent to the object

$$(...,L_{(k-2)(k-1)} \times G_{(m-2)(m-1)} \times N_{(j-2)(j-1)}, \mathcal{L} \times \Delta_X, \mathcal{L}) \tag{5.7}$$

On the other hand $(\mathcal{L} \times \Delta_X, \mathcal{L})$ can be composed by an embedded composition and gives the correspondence given in local coordinates under the isomorphism $X \cong T^*B/\Lambda$ as

$$\mathcal{T} := (x, y_1, x, y_2, x, y_3, x, y_1 + y_2 + y_3) \tag{5.8}$$

so that

$$((\underline{L},\underline{S})\hat{\otimes}(\underline{G},\underline{T}))\hat{\otimes}(\underline{N},\underline{P}) \cong (...,L_{(k-2)(k-1)} \times G_{(m-2)(m-1)} \times N_{(j-2)(j-1)}, \mathcal{T}) \tag{5.9}$$

Similarly

$$(\underline{L},\underline{S})\hat{\otimes}((\underline{G},\underline{T})\hat{\otimes}(\underline{N},\underline{P})) \cong (...,L_{(k-2)(k-1)} \times G_{(m-2)(m-1)} \times N_{(j-2)(j-1)}, \mathcal{T}) \tag{5.10}$$

which gives the isomorphisms $\alpha_{\underline{L},\underline{G},\underline{N}}$ defining the monoidal structure.

Consider $(\underline{L},\underline{S})\hat{\otimes}(\Sigma,\mathcal{O})$. It is given by

$$(...,L_{(k-2)(k-1)} \times \Sigma, \mathcal{L}) \tag{5.11}$$

But the subsequence $(L_{(k-2)(k-1)} \times \Sigma, \mathcal{L})$ is composable by an embedded composition and equals $L_{(k-2)(k-1)}$ giving the isomorphisms $\rho_{\underline{L}}$ defining the monoidal structure. Similarly one can obtain the $\lambda_{\underline{L}}$'s.

It is not difficult to check that α, ρ and λ satisfy the commutative diagrams of Definition 1.1 and so $(\hat{\otimes}, (\Sigma, \mathcal{O}))$ induces the structure of a monoidal category on $\mathrm{Don}^{\sharp}(X)$.

We can define the braiding functor as follows. Consider $(\underline{L}, \underline{S})\hat{\otimes}(\underline{G}, \underline{T})$ given by

$$(..., L_{(k-2)(k-1)} \times G_{(m-2)(m-1)}, \mathcal{L}) \tag{5.12}$$

Notice that $\mathcal{L} \subset (X \times X)^{-} \times X$ is invariant under the permutation of the first factors because of commutativity of addition on the torus fibers of X. We conclude that under the permutation symplectomorphism that takes $L_{(k-2)(k-1)} \times G_{(m-2)(m-1)}$ to $G_{(m-2)(m-1)} \times L_{(k-2)(k-1)}$ and so on for all members of the sequence, the generalized Lagrangian

$$(..., L_{(k-2)(k-1)} \times G_{(m-2)(m-1)}, \mathcal{L}) \tag{5.13}$$

goes to

$$(..., G_{(m-2)(m-1)} \times L_{(k-2)(k-1)}, \mathcal{L}) \tag{5.14}$$

and this symplectomorphism induces the braiding isomorphisms $B_{\underline{L},\underline{G}}$. These clearly satisfy $B_{\underline{L},\underline{G}} = B^{-1}_{\underline{G},\underline{L}}$ and since it's not hard to check the necessary commutative diagrams our monoidal structure is symmetric. $\square$

Chapter 6

The Donadlson-Fukaya category of a torus

Consider the 2-torus $E = \mathbb{R}^2/\mathbb{Z}^2$ equipped with the symplectic structure given by $\omega = A dx \wedge dy$, where A is then the area. Take $\Omega = \sqrt{A}(dx + idy)$, so that together they define a Calabi-Yau structure on E, and furthermore a torus fibration like the one discussed in the previous chapter. Closed Lagrangian submanifolds of E are just closed embedded curves in E. For each such curve we can apply the mean curvature flow, with the metric given by the above Calabi-Yau structure. By the properties of the mean curvature flow, if the curve is not contractible it will converge to a geodesic in its hamiltonian isotopy class. Otherwise it will converge to a point and the flow is not a hamiltonian isotopy (one cannot lift the phase function to a hamiltonian in this case). Thus in describing the Donaldson category of E, we can restrict ourselves to considering underlying Lagrangians that are geodesics for the flat metric on the torus. We discard the contractible curves whose Floer homologies are obstructed, as is usually done ([12]).

Let the **extended Donaldson category** $\mathrm{Don}^+(E)$ be the category $\mathrm{Don}(E)$ enlarged by formal sums. The functor $\hat{\otimes}$ extends to $\mathrm{Don}^+(E)$, and the main theorem of this section is

Theorem 6.0.35. *The bi-functor $\hat{\otimes}$ restricts to a bi-functor $\mathrm{Don}^+(E) \times \mathrm{Don}^+(E) \to \mathrm{Don}^+(E)$.*

Proof. Consider the d to 1 symplectic covering maps $p_d : E_d \to E$ where p_d sends (x, y) to (dx, y) and E_d denotes E equipped with $d\omega_{standard}$ (a **symplectic covering** is a map $f : (M_1, \omega_1) \to (M_2, \omega_2)$ that is a covering and satisfies $f^*\omega_2 = \omega_1$).

Lemma 6.0.36. *The pre-image under p_{kr} of the line L given by $y = \frac{n}{r}x - a$, where k is an integer and $(n, r) = 1$, is given as a disjoint union of lines L^α given by $y = knx - \frac{n}{r}\alpha - a$ where $\alpha \in \{0, ..., r-1\}$ each one of which is a k to 1 cover under p_{kr}.*

The pre-image of the line $x = b$ is kr lines $x = \frac{b+\alpha}{kr}$, where $\alpha \in \{0, ..., kr-1\}$ each one of which is isomorphic to $x = b$ under p_{kr}.

Proof. The pre-image of $y = \frac{n}{r}x - a$ under the covering $\mathbb{R}^2 \to E$ is given by lines of the form

$$y + \beta = \frac{n}{r}(x - \alpha) - a \tag{6.1}$$

where $\alpha, \beta \in \mathbb{Z}$. The pre-image of these under p_{kr} is

$$y + \beta = \frac{n}{r}(krx - \alpha) - a = knx - \frac{n}{r}\alpha - a \tag{6.2}$$

so in E_{kr} we get r curves given by projections of $y = knx - \frac{n}{r}\alpha - a$ where $\alpha \in \{0, ..., r-1\}$ each one of which is a k to 1 cover because each point has kr pre-images and there are r distinct curves.

If the curve is given by $x = b$ then the pre-image is clearly kr lines given by $x = \frac{b+\alpha}{kr}$.

$\square$

We will prove that $(L_1, \mathcal{S}_1)\hat{\otimes}(L_2, \mathcal{S}_2)$ is isomorphic to an element of $\mathrm{Don}(E)$ as a $\mathrm{Don}(E)$-module for any two objects $(L_1, \mathcal{S}_1)$ and $(L_2, \mathcal{S}_2)$ of $\mathrm{Don}(E)$ (where we suppress the local systems from the notation).

Let L_1 and L_2 be such that the sequence $(L_1 \times L_2, \mathcal{L})$ is transverse and that the projection map $\pi|_{(L_1 \times L_2) \times_{E \times E} \mathcal{L}}$ is a covering map of degree n onto an embedded Lagrangian image. We will denote this image again by $(L_1 \times L_2) \circ \mathcal{L}$. It is easy to show that the generalized intersection points of

$$(L_1 \times L_2, \mathcal{L}, T) \tag{6.3}$$

and

$$((L_1 \times L_2) \circ \mathcal{L}, T) \tag{6.4}$$

are in an n to 1 correspondence. We will show that one can indeed compute $L_1 \hat{\otimes} L_2$ in terms of $(L_1 \times L_2) \circ \mathcal{L}$ and a suitable local system on it.

We split the proof into two cases according to the following distinction. Consider the sequence $(L_1 \times L_2, \mathcal{L})$. If $L_1 \times L_2 \times \mathcal{L}$ and $\Delta_{E \times E} \times E$ intersect transversely we'll say that L_1, L_2 and $\mathcal{L}$ form a **transverse sequence**.

Lemma 6.0.37. *The lines L_1, L_2 and the Lagrangian $\mathcal{L}$ do not form a transverse sequence only when $L_1 = L_2 = F$ where F is a vertical line $x = b$.*

Proof. The tangent space $TL_1 \oplus TL_2 \oplus T\mathcal{L}$ is given by

$$(l_1, l_2, u, v_1, u, v_2, u, v_1 + v_2) \tag{6.5}$$

where $l_i \in TL_i$ and $(u, v_1, u, v_2, u, v_1 + v_2) \in T\mathcal{L}$. The tangent space $T\Delta_{E\times E} \oplus TE$ is just

$$(p, q, p, q, r) \tag{6.6}$$

where $p, q, r \in TE$. Finally let

$$(s_1, s_2, s_3, s_4, s_5) \tag{6.7}$$

be an arbitrary element of $TE \oplus TE \oplus TE \oplus TE \oplus TE$.

For a transverse sequence we need to be able to solve

$$(p + l_1, q + l_2, p + (u, v_1), q + (u, v_2), r + (u, v_1 + v_2)) = (s_1, s_2, s_3, s_4, s_5) \tag{6.8}$$

for $p, q, r, l_1, l_2, (u, v_1), (u, v_2)$ given s_1, s_2, s_3, s_4, s_5. Given l_1 and l_2, the first and second component equations determine p and q. We can pick u,v_1 and v_2 so that the third and fourth component equations are satisfied as long as $s_{3,x} - p_x = s_{4,x} - q_x$ i.e. $s_{3,x} - s_{1,x} + l_{1,x} = s_{4,x} - s_{2,x} + l_{2,x}$ (where subscript x denotes the base direction component of the vector). Finally we can pick r freely so that the last component equation is satisfied.

Thus as long as $l_{2,x} - l_{1,x}$ can be any real number for a choice of (l_1, l_2) the sequence is transverse, which is true whenever at least one of the tangent spaces to lines L_1 and L_2 at generalized intersection points has a horizontal component, which is in turn true for all pairs of lines L_1 and L_2 except when they are both equal to a vertical line $x = b$. $\square$

So let us start with the transverse case.

6.0.4 Case 1

Let L_1 and L_2 be two lines (geodesics for the flat metric on the torus) given by $y = \frac{n_1}{r_1}x - a_1$ and $y = \frac{n_2}{r_2}x - a_2$. Notice that in this case the projection map $\pi|_{(L_1 \times L_2) \times_{E \times E} \mathcal{L}}$ is a degree w covering, where $w = \gcd(r_1, r_2)$. This is so for the following reason: notice that the first line intersects a vertical line in r_1 equally spaced points, and the second one in r_2 equally spaced points. Over each point (x, z) of $(L_1 \times L_2) \circ \mathcal{L}$ there will be all the points (x, y_1, x, y_2, x, z) in $(L_1 \times L_2) \times_{E \times E} \mathcal{L}$ such that $(x, y_1, x, y_2) \in L_1 \times L_2$ and $y_1 + y_2 = z$, and there are precisely w of those.

Let T be a line given by $y = \frac{n_3}{r_3}x - a_3$. Take d to be $r_1 r_2 r_3$. To analyze the quilts involved, consider the covering $p_d \times p_d \times p_d : E_d^- \times E_d^- \times E_d \to E^- \times E^- \times E$. The pre-image of $\mathcal{L}$ consists of disjoint tori given by

$$\mathcal{L}_{i,j} := (x + \frac{i}{d}, y_1, x + \frac{j}{d}, y_2, x, y_1 + y_2) \tag{6.9}$$

where $0 \leq i, j \leq d - 1$.

Equip $E_d^- \times E_d^- \times E_d$ with pullback almost-holomorphic structures. Let $\underline{u}$ be a quilted strip with seam conditions $(L_1 \times L_2, \mathcal{L}, T)$ (where we write T for T^T to simplify notation).

Lemma 6.0.38. *The quilted strips $\underline{u}$ are in a d^3 to 1 correspondence with the quilted strips with seams in the pre-image Lagrangians $(L_1^\alpha \times L_2^\beta, \mathcal{L}_{i,j}, T^\gamma)$, where $0 \leq i, j \leq$*

$d-1$, $0 \le \alpha \le r_1 - 1$, $0 \le \beta \le r_2 - 1$, $0 \le \gamma \le r_3 - 1$.

Proof. A quilted strip $\underline{u}$ consists of a pair (u_1, u_2) from generalized intersection point $< x_- >= (n_{1,-} \times n_{2,-}, n_{3,-})$ to generalized intersection point $< x_+ >= (n_{1,+} \times n_{2,+}, n_{3,+})$. Pick one of the d^3 lifts of $< x_- >$ under $p_d \times p_d \times p_d$. By definition this is a generalized intersection point for one of the $(L_1^\alpha \times L_2^\beta, \mathcal{L}_{i,j}, T^\gamma)$ and it determines a lift of (u_1, u_2) to a quilted strip with these lifted seams (just lift u_1 and u_2 individually and by definition they will form such a quilted strip). Since every quilted strip with seam conditions $(L_1^\alpha \times L_2^\beta, \mathcal{L}_{i,j}, T^\gamma)$ projects to one with conditions $(L_1 \times L_2, \mathcal{L}, T)$ this finishes the proof. $\square$

Now by Lemma 4.6, the quilted strips with seam conditions in $(L_1^\alpha \times L_2^\beta, \mathcal{L}_{i,j}, T^\gamma)$ are in a 1-1 correspondence with quilted strips with seam conditions in $(C_{\alpha-i,\beta-j}, T^\gamma)$. Where

$$C_{\alpha-i,\beta-j} := (L_1^\alpha \times L_2^\beta) \circ \mathcal{L}_{i,j} \tag{6.10}$$

and $C_{k,l}$ is given by

$$y = (n_1 \frac{d}{r_1} + n_2 \frac{d}{r_2})x - \frac{n_1}{r_1}k - \frac{n_2}{r_2}l - a_1 - a_2 \tag{6.11}$$

There are p different $C_{k,l}$ for α,β,i,j varying, where $p = \frac{r_1 r_2}{w}$. This is because counting the number of different $C_{k,l}$ is the same as counting the number of $n_1 r_2 k + n_2 r_1 l \mod r_1 r_2$, which is just $\frac{r_1 r_2}{\gcd(\gcd(n_1 r_2, n_2 r_1), r_1 r_2)} = \frac{r_1 r_2}{\gcd(\gcd(n_1, n_2)w, r_1 r_2)} = p$. There are $r_1 r_2 d^2$ different seam conditions $(L_1^\alpha \times L_2^\beta, \mathcal{L}_{i,j})$ so the quilted strips with these seam conditions are in a $d^2 w$ to 1 correspondence with quilted strips with conditions $(C_{k,l}, T^\gamma)$.

The curves $C_{k,l}$ are the pre-images of the curves $R_{k,l}$ by p_d, where $R_{k,l}$ is

$$y = (\frac{n_1}{r_1} + \frac{n_2}{r_2})x - \frac{n_1}{r_1}k - \frac{n_2}{r_2}l - a_1 - a_2 \tag{6.12}$$

There are q of these curves, where $q = \gcd(p, \frac{n_1 r_2}{w} + \frac{n_2 r_1}{w})$, because their number is the number of values of $n_1 r_2 l_1 + n_2 r_1 l_2 \bmod \gcd(r_1 r_2, n_1 r_2 + n_2 r_1) = qw$. The number of these is $\frac{qw}{\gcd(\gcd(n_1 r_2, n_2 r_1), qw)}$. But this is $\frac{q}{\gcd(\gcd(n_1, n_2), q)}$, and $\gcd(n_1, n_2)$ and q are co-prime, so the final answer is q. Each of these q curves has $\frac{p}{q}$ pre-images that all together give the p curves $R_{k,l}$.

Using the same argument as in Lemma 6.4 we conclude that the strips with boundary conditions $(C_{k,l}, T^\gamma)$ are in a d to 1 correspondence with strips with boundary conditions $(R_{k,l}, T)$. But $\oplus_{k,l} R_{k,l}$ is precisely $(L_1 \times L_2) \circ \mathcal{L}$ for

$$y = \frac{n_1}{r_1}x - a_1 \tag{6.13}$$

and

$$y = \frac{n_2}{r_2}x - a_2 \tag{6.14}$$

Thus we can conclude that the quilted strips with seam conditions in $(L_1 \times L_2, \mathcal{L}, T)$ are in a w to 1 correspondence with the quilted strips with seam conditions in $(L_1 \times L_2) \circ \mathcal{L}$.

Since all the generalized intersection points of $(L_1 \times L_2, \mathcal{L})$ and T have the same index, we can find a $\mathcal{S}$ on $(L_1 \times L_2) \circ \mathcal{L}$ that will give an isomorphism

$$CF^*((L_1 \times L_2, \mathcal{L}), T) \cong CF^*((L_1 \times L_2) \circ \mathcal{L}, T) \tag{6.15}$$

and an isomorphism

$$HF^*(L_1 \times L_2, \mathcal{L}, T) \cong (HF^*((L_1 \times L_2) \circ \mathcal{L}, T)) \tag{6.16}$$

where we suppressed the local systems from the equations for ease of writing.

Take the pullback local systems $(\mathcal{S}_1^\alpha \boxtimes \mathcal{S}_2^\beta, \mathcal{O}, \mathcal{T}^\gamma)$ on $(L_1^\alpha \times L_2^\beta, \mathcal{L}_{i,j}, T^\gamma)$, where $\mathcal{T}$ is the local system on T. On $(C_{\alpha-i,\beta-j}, T^\gamma)$ under the composition this goes to $((\mathcal{S}_1^\alpha \boxtimes \mathcal{S}_2^\beta) \circ \mathcal{O}, \mathcal{T}^\gamma)$.

On the other hand take the rank w local system on $(L_1 \times L_2) \circ \mathcal{L}$ given by $\pi_* i^*((L_1 \times L_2) \circ \mathcal{L})$ (which is well defined for a covering map π). We will call this local system $(\mathcal{S}_1 \boxtimes \mathcal{S}_2) \circ \mathcal{O})$. Pulling this back to $(C_{k,l}, T^\gamma)$ one gets that

$$\oplus_{\alpha,\beta,i,j}(C_{\alpha-i,\beta-j}, (\mathcal{S}_1^\alpha \boxtimes \mathcal{S}_2^\beta) \circ \mathcal{O}) \tag{6.17}$$

and

$$\oplus_{k,l}(C_{k,l}, p_d^*((\mathcal{S}_1 \boxtimes \mathcal{S}_2) \circ \mathcal{O})) \tag{6.18}$$

have the same Floer homologies with T^γ. Thus $(\mathcal{S}_1 \boxtimes \mathcal{S}_2) \circ \mathcal{O})$ is exactly the local system we are looking for.

We can rephrase this in terms of $\mathrm{Don}(E)$ modules as a set of isomorphisms $\tilde{\eta}_T$: $(L_1 \hat{\otimes} L_2)(T) \cong ((L_1 \times L_2) \circ \mathcal{L})(T)$ (where the local systems are suppressed from notation). We want to claim that η_T induce an isomorphism of modules.

Remark 6.0.39. *To obtain such isomorphisms one could just notice that all of the Lagrangians involved are special. Equip $E \times E \times E$ with the product Calabi-Yau structure using $-(\omega \oplus \omega) \oplus \omega$ and $\overline{(\Omega \wedge \Omega)} \wedge \Omega$ (giving the complex structure $-(J \times J) \times J$).*

The Lagrangian $\mathcal{L}$ is special for the product Calabi-Yau structure on $(E \times E)^- \times E$ with constant phase function -1.

To see this decompose E^3 as $B \times T \times B \times T \times B \times T$ with coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$. Consider the embedding of $B \times T \times T$ into E^3 that gives $\mathcal{L}$ so that the coordinates on

$B \times T \times T$ are (x, y_1, y_2). Restricting the holomorphic volume form gives

$$\Omega_{E^3}|_{\mathcal{L}} = A^{\frac{3}{2}}(dx - idy_1) \wedge (dx - idy_2) \wedge (dx + idy_1 + idy_2) = -3A^{\frac{3}{2}} dx \wedge dy_1 \wedge dy_2 \quad (6.19)$$

The metric is given by $A(dx_1^2 + dy_1^2 + dx_2^2 + dy_2^2 + dx_3^2 + dy_3^2)$, and the volume form of the restriction is $3A^{\frac{3}{2}} dx \wedge dy_1 \wedge dy_2$.

As a special Lagrangian $\mathcal{L}$ admits a natural grading by picking a real number such that $\exp(2\pi i\alpha) = (-1)^2 = 1$. We consider $\mathcal{L}$ as a graded Lagrangian submanifold with the grading given by $\alpha = 0$.

Suppose now for example that we are considering three special Lagrangians L_1, L_2 and L_3 in E with phases α_1, α_2, and α_3. Then one can show that Lagrangian $L_1 \times L_2 \times L_3$ is special in $E \times E \times E$ and has the phase function $\exp \pi i(-(\alpha_1 + \alpha_2) + \alpha_3)$.

However, the discussion we gave before is more general and that the choice of η_T is canonical and enables one to give a $\mathrm{Don}(E)$-module isomorphism which we discuss next. Still we use these observations when we make remarks on the indices of generalized intersection points we consider.

To do this, let S be a line given by $y = \frac{n_4}{r_4}x - a_4$. Let $\underline{v}$ be a quilted half pair of pants with seam conditions $(L_1 \times L_2, \mathcal{L})$, T and S.

Lemma 6.0.40. *The quilted half pairs of pants $\underline{v}$ are in a d^3 to 1 correspondence with the quilted half pairs of pants with seam conditions $(L_1^\alpha \times L_2^\beta, \mathcal{L}_{i,j})$, T^γ, S^δ where $0 \leq i, j \leq d-1$, $0 \leq \alpha \leq r_1 - 1$, $0 \leq \beta \leq r_2 - 1$, $0 \leq \gamma \leq r_3 - 1$, $0 \leq \delta \leq r_4 - 1$.*

Proof. A quilted half-pair of pants $\underline{v}$ consists of a pair of maps (v_1, v_2), where v_1 is a holomorphic strip in $E \times E$ and v_2 is a pair of pants in E from $< x_- > \otimes p$, where $< x_- >= (n_{1,-} \times n_{2,-}, n_{3,-})$ is a generalized intersection of $(L_1 \times L_2, \mathcal{L}, T)$ and p is an intersection point of T and S, to $< x_+ >$. Pick one of the d^3 lifts of $< x_- >$ under $p_d \times p_d \times p_d$. This determines a lift of (v_1, v_2), which is by definition a half-pair of pants with seam conditions $(L_1^\alpha \times L_2^\beta, \mathcal{L}_{i,j})$, T^γ, S^δ. Conversely every such half-pair of pants projects to a half-pair of pants with conditions $(L_1 \times L_2, \mathcal{L})$, T and S which finishes the proof. $\square$

On the other hand quilted half-pair of pants with seam conditions $(L_1^\alpha \times L_2^\beta, \mathcal{L}_{i,j})$, T^γ, S^δ are in a $1-1$ correspondence with half-pair of pants with seam conditions $C_{\alpha-i,\beta-j}$, T^γ and S^δ by Theorem 4.7. Completing the argument like for quilted half strips we conclude that the quilted half-pairs of pants with seam conditions

$$(L_1 \times L_2, \mathcal{L}), T, S \tag{6.20}$$

are in a w to 1 correspondence with half-pairs of pants with boundary conditions

$$(L_1 \times L_2) \circ \mathcal{L}, T, S \tag{6.21}$$

In terms of Don(E)-modules this says (by the part of Theorem 4.7 concerning the intertwining property of the relative invariants defined by these) that the maps η_T are natural transformations.

Without loss of generality one can take T and S to be any combination of possible lines that are Lagrangians of Don(E) (possibly doing a small hamiltonian perturbation to achieve transversality, but that does not change the argument). Similarly one can analogously argue for any combination of L_1 and L_2 that form a transverse sequence with $\mathcal{L}$, the non-transverse case being the subject of case 2. Thus, in the transverse case we have proved $(L_1 \hat{\otimes} L_2, \mathcal{S}_1 \hat{\otimes} \mathcal{S}_2) \cong ((L_1 \times L_2) \circ \mathcal{L}, (\mathcal{S}_1 \boxtimes \mathcal{S}_2) \circ \mathcal{O})$ as Don(E)-modules.

6.0.5 Case 2

Suppose that L_1 and L_2 both have the underlying Lagrangian given by a vertical line F given by equation $x = b$. We will prove that then either $L_1 \hat{\otimes} L_2 \cong F[-1] \oplus F$ (with local systems to be explained) or $L_1 \hat{\otimes} L_2 \cong 0$ as $\mathrm{Don}(E)$-modules.

The tensor product in $\mathrm{Don}^\sharp(E)$ is given by $L_1 \hat{\otimes} L_2 = (L_1 \times L_2, \mathcal{L})$. This is not transverse so let us deform L_1 with a small hamiltonian deformation so that it has two intersection points with L_2. Let T be a test object of the form

$$y = nx - a \tag{6.22}$$

where $n \in \mathbb{Z}$ and $a \in \mathbb{R}$.

Lemma 6.0.41. *The composition $\mathcal{L} \circ T$ is embedded.*

Proof. Notice that the first two components of $(\mathcal{L} \times T) \times_E (E \times E \times \Delta_E)$ determine the rest, and so the map to $\mathcal{L} \circ T$ is an embedding. □

Consider the linear transformation

$$A = \begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix} \tag{6.23}$$

Let ϕ be the affine transformation $(x, y) \mapsto (x, y + a)$.

Lemma 6.0.42. *The transformation $\psi = id \times (A \circ \phi)$ is a symplectomorphism $E^- \times E \to (E \times E)^-$ that takes the diagonal Δ_E to $\mathcal{L} \circ T$.*

Proof. The transformation A takes $\mathbb{Z}^2$ to itself in $\mathbb{R}^2$ and so descends to E. $A^{-1} = A$, which also descends so it induces a linear isomorphism. In coordinates it takes (x, y) to $(x, nx - y)$ so it induces a symplectomorphism from $dx \wedge dy \to -dx \wedge$

dy. The transformation ϕ is obviously a symplectomorphism so $id \times (\phi \circ A)$ is a symplectomorphism $E^- \times E \to (E \times E)^-$.

Taking the action on Δ_E in coordinates

$$(x, y, x, y) \to (x, y, x, nx - y - a) \tag{6.24}$$

This is precisely $\mathcal{L} \circ T$. □

So we get that

$$HF^*(L_1 \times L_2, \mathcal{L}, T^\vee) \cong HF^*(L_1 \times L_2, \mathcal{L} \circ T^\vee) \cong HF^*(\psi(L_1 \times L_2^T), \psi(\Delta_E)) \tag{6.25}$$

where the last isomorphism uses the fact that ϕ fixes L_2. Thus

$$HF^*(L_1 \hat{\otimes} L_2, T) \cong HF^*(L_1 \times L_2^T, \Delta_E) \cong HF^*(L_1, L_2) \tag{6.26}$$

This chain of isomorphisms implies that the only quilted strips between the two generalized intersection points of $(L_1 \times L_2, \mathcal{L})$ and T are the ones coming from the discs between L_1 and L_2.

Now we need to reintroduce the local systems into the computation. We can assume that our objects given by $(L_1, \mathcal{S}_1)$ and $(L_2, \mathcal{S}_2)$ are indecomposable with no loss of generality. By definition we can then specify their local systems using holonomies $\exp(2\pi i b_i 1_{V_i} + N_i)$ where N_i are cyclic nilpotent endomorphisms on fibers V_i of S_i, and b_i are real numbers. If $b_1 \neq b_2$ the Floer homology $HF^*((L_1, \mathcal{S}_1), (L_2, \mathcal{S}_2))$ vanishes by definition (it is zero when there is no common eigenvalue for holonomies), and so tracing back the differential on

$$CF^*(L_1 \hat{\otimes} L_2, T) \cong CF^{*-1}((L_1 \times L_2) \circ \mathcal{L}, T) \oplus CF^*((L_1 \times L_2) \circ \mathcal{L}, T) \tag{6.27}$$

equals zero and so

$$HF^*(L_1\hat{\otimes}L_2,T)\cong HF^{*-1}((L_1\times L_2)\circ\mathcal{L},T)\oplus HF^*((L_1\times L_2)\circ\mathcal{L},T) \tag{6.28}$$

If $b_1=b_2$ one can compute ([12]) that $HF^*((L_1,\mathcal{S}_1),(L_2,\mathcal{S}_2))$ is isomorphic to V_i for the i such that V_i is of smaller rank, which will correspond to taking the local system on $(L_1\times L_2)\circ\mathcal{L}$ to be S_i. We denote this local system by $(\mathcal{S}_1\boxtimes\mathcal{S}_2)\circ\mathcal{O}$ (it is the cohomology of $\pi_*i^*(\mathcal{S}_1\boxtimes\mathcal{S}_2)$ considered as a complex of local systems with a differential induced by the above discussion).

To establish that the same holds for other test objects, let T be given by

$$y=\frac{n}{r}x-a \tag{6.29}$$

and consider the cover map p_r where p is as in the previous proposition. We again have

$$CF^*(L_1\hat{\otimes}L_2,T)\cong CF^{*-1}((L_1\times L_2)\circ\mathcal{L},T)\oplus CF^*((L_1\times L_2)\circ\mathcal{L},T) \tag{6.30}$$

By the same reasoning as in Section 1 we get that quilted strips with seam conditions $(L_1\hat{\otimes}L_2,T)$ are in correspondence with quilted strips with seam conditions in $(L_1^\alpha\times L_2^\beta,\mathcal{L}_{i,j},T^\gamma)$, where L_1^α, L_2^β, $\mathcal{L}_{i,j}$ and T^γ are as before.

The sequence $(L_1^\alpha\times L_2^\beta,\mathcal{L}_{i,j},T^\gamma)$ has generalized intersection points when $\alpha+i=\beta+j$ and for each such we get 2 quilts by the $r=1$ part of the argument. Thus we get $2r^4$ quilts for all $(L_1^\alpha\times L_2^\beta,\mathcal{L}_{i,j},T^\gamma)$, and so we originally had $2r$ quilts such that

$$CF^*(L_1\hat{\otimes}L_2,T)\cong CF^{*-1}((L_1\times L_2)\circ\mathcal{L},T)\oplus CF^*((L_1\times L_2)\circ\mathcal{L},T) \tag{6.31}$$

becomes either

$$HF^*(L_1\hat{\otimes}L_2,T)\cong HF^{*-1}((L_1\times L_2)\circ\mathcal{L},T)\oplus HF^*((L_1\times L_2)\circ\mathcal{L},T) \tag{6.32}$$

or 0 as before.

Finally let $T = F$, the line $x = b$. In this case $\mathcal{L} \circ T \cong F \times F$ so we get that

$$HF^*(L_1 \hat{\otimes} L_2, T) \cong HF^*(F \times F, F \times F) \cong HF^*(F, F) \otimes HF^*(F, F) \tag{6.33}$$

But

$$HF^{*-1}((L_1 \times L_2) \circ \mathcal{L}, T) \oplus HF^*((L_1 \times L_2) \circ \mathcal{L}, T) \cong HF^{*-1}(F, F) \oplus HF^*(F, F) \tag{6.34}$$

in this case so we get the expected answer. As we discussed composition is natural with respect to quilted triangles so we conclude that in terms of $\mathrm{Don}(E)$-modules

$$L_1 \hat{\otimes} L_2 \cong ((L_1 \times L_2) \circ \mathcal{L})[-1] \oplus ((L_1 \times L_2) \circ \mathcal{L}) \tag{6.35}$$

with local systems described, or 0 depending on the eigenvalues of holonomies for $\mathcal{S}_1$ and $\mathcal{S}_2$.

This then finishes Case 2, and ends the proof.

$\square$

Chapter 7

Mirror symmetry for the elliptic curve as a monoidal equivalence

In this chapter we discuss mirror symmetry for a torus E and how the new tensor product, as applied to E, mirrors the one on its dual. Let E_{dual} be the elliptic curve with modular parameter $\tau = \rho$. In [12] we are given a functor Φ that induces an equivalence between the category $\mathrm{Don}^+(E)$ and the derived category of coherent sheaves on E_{dual}. Let us review the construction of this functor, by starting with a short review of the theory of coherent sheaves on an elliptic curve.

We take the following result as our starting point:

Theorem 7.0.43. *[6] Let X be a smooth projective curve. The full subcategory of $D^b_{coh}(X)$ formed by all objects that are finite direct sums of objects of the form $A[n]$, where A is either an indecomposable torsion sheaf supported at a point or an indecomposable vector bundle, is equivalent to $D^b_{coh}(X)$.*

In case of an elliptic curve, indecomposable torsion sheaves and vector bundles can be further classified. Identify X with $\mathbb{C}/\Lambda$ where Λ is a lattice generated by 1 and τ. Every line bundle on X is given as

$$(\mathbb{C} \times \mathbb{C})/\Lambda \tag{7.1}$$

where the action of the lattice is given by 1 acting as identity and

$$\tau(z, v) = (z + \tau, \phi(z)v) \tag{7.2}$$

where ϕ is some function from $\mathbb{C}$ to $\mathbb{C}^*$. Denote a line bundle constructed in this way by $L(\phi)$. We will distinguish the line bundle L_0 given by the function

$$\phi_0(z) = \exp(-\pi i\tau - 2\pi i z) \tag{7.3}$$

Every line bundle can be represented as $L = t_x^* L_0 \otimes L_0^{n-1}$, for some translation t_x and some integer n, and so is given by

$$\phi(z) = t_x^* \phi_0 \cdot \phi_0^{n-1} \tag{7.4}$$

Every vector bundle is given as $(\mathbb{C} \times V)/\Lambda$, where V is a vector space and the action of Λ is given by 1 acting as identity and

$$\tau(z, v) = (z + \tau, A(z)v) \tag{7.5}$$

where A is a function from $\mathbb{C}$ to $GL(V)$. We denote such bundles by $F(V, A)$. These can be classified as well, like in work of Atiyah [1] and later Oda [11]. Consider the natural isogeny map $\pi_r : X_{r\tau} \to X_\tau$ where r is an integer, and X_τ denotes the elliptic curve with modular parameter τ.

Theorem 7.0.44 (Atiyah, Oda). *Any indecomposable vector bundle on X of rank and degree (rk, nk) $(gcd(r, n) = 1)$ can be written in the form $\pi_{r*}(L(\phi) \otimes F(V, \exp(N)))$,* *where $L(\phi)$ is a uniquely given degree n vector bundle and N is a cyclic nilpotent endomorphism of a k-dimensional vector space V.*

Torsion sheaves supported at a point are easier to describe. They are determined completely by the point of support x, the vector space of global sections V and the action of $(z-x)$ (considered as a generator of $\mathcal{O}_x$) on V that is given by a nilpotent matrix $N \in End(V)$. We denote such sheaves by $S(x, V, N)$.

Now we can describe the action of Φ on the objects we discussed. If $A = \pi_{r*}(L(\phi)\otimes F(V, \exp N))$ with $\phi = t_{a\tau+b}\phi_0 \cdot \phi_0^{n-1}$ the image is

$$\Phi(A) = (\Lambda, \alpha, M) \tag{7.6}$$

where Λ is the line with slope $\frac{n}{r}$ and x intercept $\frac{ar}{n}$, the phase α is the unique real number with $-\frac{1}{2} < \alpha \leq \frac{1}{2}$ and $\exp(i\pi\alpha) = \frac{1+i\frac{n}{r}}{\sqrt{1+(\frac{n}{r})^2}}$, and the local system $M \in GL(V^{\oplus d})$ is the pushforward by the degree d cover of the local system with monodromy $\exp(-2\pi i b 1_V + N)$.

If $A = S(a\tau + b, V, N)$ the image is

$$\Phi(A) = (\Lambda, \frac{1}{2}, \exp(2\pi i b 1_V + N)) \tag{7.7}$$

where Λ is the vertical line through $-a$.

We are ready to prove the following theorem

Theorem 7.0.45. *Let E be a 2-torus with complexified Kähler parameter $\rho = iA+b$, where A is its area. Let E_{dual} be an elliptic curve with modular parameter $\tau = \rho$, and let D^{alg} be the bounded derived category of coherent sheaves on E_{dual} considered as a monoidal category with monoidal structure induced by the standard tensor product. Let*

D^{symp} be the extended Donaldson category of E $\mathrm{Don}^+(E)$ equipped with the monoidal structure induced by $\hat{\otimes}$. Then D^{alg} and D^{symp} are equivalent as monoidal categories.

Proof. Let us show that Φ is a monoidal functor. To do this we will split our discussion into three cases. In each case we need to compute the Oda representation for the tensor products that we need to use the mirror functor and then we will compare with the mirror tensor product.

Case 1

Suppose that we are looking at $A \otimes B$ where

$$A = \pi_{r*}(L_1 \otimes F(V_1, \exp(N_1))) \tag{7.8}$$

and

$$B = \pi_{s*}(L_2 \otimes F(V_2, \exp(N_2))) \tag{7.9}$$

with $L_1 = t^*_{x_1} L_0 \otimes L_0^{n-1}$ and $L_2 = t^*_{x_2} L_0 \otimes L_0^{m-1}$ and V_1, V_2 of ranks k_1 and k_2 respectively (remember $gcd(n, r) = gcd(m, s) = 1$). For notational convenience we will again use E_τ to denote the elliptic curve with modular parameter τ. Notice the following cartesian square

$$\begin{tikzcd} E \arrow[r, "\mathrm{l_r}"] \arrow[d, "\mathrm{l_s}"'] & E_{r\tau} \arrow[d, "\pi_{\mathrm{r}}"] \\ E_{s\tau} \arrow[r, "\pi_{\mathrm{s}}"] & E_\tau \end{tikzcd} \tag{7.10}$$

where $E \cong E_{p\tau} \times \mathbb{Z}/d\mathbb{Z}$, where $d = gcd(r, s)$ and $p = \frac{rs}{d}$. Denote by $l_{r,\nu}$ the composition of the isogeny $E_{p\tau} \to E_{r\tau}$ with the translation by $\nu\tau$ on $E_{r\tau}$, and by $l_{s,\nu}$ the isogeny $E_{p\tau} \to E_{s\tau}$. These are just the maps obtained from mapping $E_{p\tau}$ to the ν place in $E_{p\tau} \times \mathbb{Z}/d\mathbb{Z}$ and composing with l_r and l_s (this depends on the choice of identification of E with $E_{p\tau} \times \mathbb{Z}/d\mathbb{Z}$ and this choice is the most convenient).

We want to compute

$$A \otimes B = \pi_{r*}(L_1 \otimes F(V_1, \exp(N_1))) \otimes \pi_{s*}(L_2 \otimes F(V_2, \exp(N_2))) \tag{7.11}$$

in standard form so we could find its mirror. Since $\pi_r^* F(V_1, \exp(\frac{1}{r}N_1)) = F(V_1, \exp(N_1))$ and similarly for s, we can use the projection formula to rewrite

$$A \otimes B = (\pi_{r*}L_1 \otimes \pi_{s*}L_2) \otimes (F(V_1, \exp(\frac{1}{r}N_1)) \otimes F(V_2, \exp(\frac{1}{s}N_2))) \tag{7.12}$$

Let us compute the part involving line bundles

$$\begin{aligned} \pi_{r*}L_1 \otimes \pi_{s*}L_2 &= \pi_{r*}(L_1 \otimes \pi_r^* \pi_{s*} L_2) \\ &= \pi_{r*}(L_1 \otimes l_{r*} l_s^* L_2) \\ &= \pi_{r*} l_{r*}(l_r^* L_1 \otimes l_s^* L_2) \\ &= \pi_*(l_r^* L_1 \otimes l_s^* L_2) \end{aligned}$$

where $\pi : E \to E_\tau$ is the composition map (using base change $\pi_r^* \pi_{s*} = l_{r*} l_s^*$).

We can decompose this into factors as

$$\pi_{r*}L_1 \otimes \pi_{s*}L_2 = \oplus_\nu \pi_{p*}(l_{r,\nu}^* L_1 \otimes l_{s,\nu}^* L_2) \tag{7.13}$$

where $\pi_{p*} : E_{p\tau} \to E_\tau$ is as before. We will use the following two Lemmas that we prove here for convenience.

Lemma 7.0.46. $t_y^*(t_x^* L_0 \otimes L_0^{n-1}) = t_{x+ny}^* L_0 \otimes L_0^{n-1}$

Proof. Notice that for every two points x and y we have

$$t_x^* L_0 \otimes t_y^* L_0 = t_{x+y}^* L_0 \otimes L_0 \tag{7.14}$$

by the Theorem of the Square [10]. □

Lemma 7.0.47. *Let $\pi_k : E_{k\tau} \to E_\tau$ be the k-covering and $a, b \in \mathbb{R}$.*

$$\pi_k^*(t_{a\tau+b}^* L_0 \otimes L_0^{n-1}) = t_{ak\tau+kb}^* L_0 \otimes L_0^{kn-1} \tag{7.15}$$

Proof. For any line bundle L let $m_L : E_\tau \to \hat{E}_\tau$ be the map taking $x \in E_\tau$ to the line bundle $t_x^* L \otimes L^{-1}$ in the dual curve. The following diagram commutes

$$\begin{tikzcd} E_{k\tau} \arrow[r, "m_{\pi_k^* L}"] \arrow[d, "\pi_k"'] & \hat{E}_{k\tau} \\ E_\tau \arrow[r, "m_L"] & \hat{E}_\tau \arrow[u, "\pi_k^*"'] \end{tikzcd} \tag{7.16}$$

which is the same as saying

$$t_z^*(\pi_k^* L) \otimes (\pi_k^* L)^{-1} = \pi_k^*(t_{\pi_k(z)}^* L \otimes L^{-1}) \tag{7.17}$$

It is easy to compute that

$$\pi_k^* L_0 = L_0^k \tag{7.18}$$

so using L_0 as L and the previous lemma we get

$$t_{kz}^* L_0 \otimes L_0^{-1} = \pi_k^*(t_{\pi_k(z)}^* L_0 \otimes L_0^{-1}) \tag{7.19}$$

This implies the above for degree 0 bundles and analogously we conclude for all degrees. □

Let $x_1 = a_1 r\tau + b_1$ and $x_2 = a_2 s\tau + b_2$. Then

$$l_{r,\nu}^* L_1 = t_{(a_1+\frac{n\nu}{r})p\tau+\frac{s}{d}b_1}^* L_0 \otimes L_0^{\frac{s}{d}n-1} \tag{7.20}$$

and

$$l_{s,\nu}^* L_2 = t_{a_2 p\tau+\frac{r}{d}b_2}^* L_0 \otimes L_0^{\frac{r}{d}m-1} \tag{7.21}$$

so

$$\pi_{r*}L_1 \otimes \pi_{s*}L_2 = \oplus_\nu \pi_{p*}(t^*_{(a_1+a_2+\frac{n\nu}{r})p\tau+\frac{s}{d}b_1+\frac{r}{d}b_2} L_0 \otimes L_0^{\frac{s}{d}n+\frac{r}{d}m-1}) \tag{7.22}$$

For the other part of the tensor product computation notice that

$$F(V_1, \exp(\frac{1}{r}N_1)) \otimes F(V_2, \exp(\frac{1}{s}N_2)) = F(V_1 \otimes V_2, \exp(\frac{1}{r}N_1 \otimes 1 + 1 \otimes \frac{1}{s}N_2)) \tag{7.23}$$

We can decompose $V_1 \otimes V_2$ (Lemma 21 in [1]) so that $\exp(\frac{1}{r}N_1 \otimes 1 + 1 \otimes \frac{1}{s}N_2)$ is cyclic unitary on each subspace to get that $(V_1 \otimes V_2, \exp(\frac{1}{r}N_1 \otimes 1 + 1 \otimes \frac{1}{s}N_2))$ decomposes as

$$(W_{k_1-k_2+1}, O_{k_1-k_2+1}) \oplus (W_{k_1-k_2+3}, O_{k_1-k_2+3}) \oplus \ldots \oplus (W_{k_1+k_2-1}, O_{k_1+k_2-1}) \tag{7.24}$$

where W's sum up to $V_1 \otimes V_2$ and O's are restrictions of $\exp(\frac{1}{r}N_1 \otimes 1 + 1 \otimes \frac{1}{s}N_2))$ to the W's. Thus $F(V_1, \exp(\frac{1}{r}N_1)) \otimes F(V_2, \exp(\frac{1}{s}N_2))$ decomposes as

$$F(W_{k_1-k_2+1}, O_{k_1-k_2+1}) \oplus F(W_{k_1-k_2+3}, O_{k_1-k_2+3}) \oplus \ldots \oplus F(W_{k_1+k_2-1}, O_{k_1+k_2-1}) \tag{7.25}$$

where we assume $k_1 \geq k_2$. The full tensor product is then

$$A \otimes B = \oplus_{\nu,i} \pi_{p*}(t^*_{(a_1+a_2+\frac{n\nu}{r})p\tau+\frac{s}{d}b_1+\frac{r}{d}b_2} L_0 \otimes L_0^{\frac{s}{d}n+\frac{r}{d}m-1} \otimes F(W_{k_1-k_2+2i+1}, O^p_{k_1-k_2+2i+1})) \tag{7.26}$$

where we've again used the projection formula and thus the p-th powers on O's.

The mirror map takes this object to

$$\Phi(A \otimes B) = \oplus_{\nu,i}(\Lambda_\nu, \alpha, M_{\nu,i}) \tag{7.27}$$

where Λ_ν is the line with slope $\frac{1}{r}n + \frac{1}{s}m$ and x intercept $\frac{a_1+a_2+\frac{n\nu}{r}}{\frac{1}{r}n+\frac{1}{s}m}$, the number $-\frac{1}{2} < \alpha \leq \frac{1}{2}$ is given by the slope, and the local systems $M_{\nu,i}$ are in $GL(W^{\oplus q}_{k_1-k_2+2i+1})$ where $q = gcd(p, \frac{s}{d}n + \frac{r}{d}m)$. Since the local systems will not come decomposed in our

Lagrangian calculation we can sum over i. So if we denote by M_1 the monodromy of $\Phi(A)$ and M_2 the monodromy of $\Phi(B)$ we get the $q-th$ power of $(V_1 \otimes V_2, M_1^{\frac{s}{d}} \otimes M_2^{\frac{r}{d}})$

Let us compute $\Phi(A)\hat{\otimes}\Phi(B)$. The Lagrangian L_1 underlying $\Phi(A)$ is a line of slope $\frac{n}{r}$ starting at $\frac{a_1 r}{n}$ and for $\Phi(B)$ it is a line L_2 of slope $\frac{m}{s}$ starting at $\frac{a_2 s}{m}$. Remembering the last chapter, the line underlying the tensor product is given as $(L_1 \times L_2) \circ \mathcal{L}$, which is just the set of curves R_{l_1,l_2}

$$y = (\frac{n}{r} + \frac{m}{s})x - (\frac{n}{r}l_1 + \frac{m}{s}l_2) + a_1 + a_2 \tag{7.28}$$

for integers l_1 and l_2. Let us show this is the same family of lines as the one given by $\Phi(A \otimes B)$. The equivalent pairs $(l_1, l_2) \sim (l_1 + ur, l_2 + vs)$ where $u, v \in \mathbb{Z}$ give the same curve, and the pairs $(l_1, l_2) \sim (l_1 + w, l_2 + w)$ where $w \in \mathbb{Z}$ also. We can use the second relation to get rid of the second parameter to get $(l_1, l_2) \sim (\nu, 0)$ where $\nu \in \mathbb{Z}$ is free and then the first relation gives $(\nu, 0) \sim (\nu + ur + vs, 0)$ which by definition of d gives the lines can be written as

$$y = (\frac{n}{r} + \frac{m}{s})x - \frac{n}{r}\nu - a_1 - a_2 \tag{7.29}$$

where ν ranges from 0 to d.

Notice that this is not a family of distinct curves, we just wanted to prove that the two families coincide. The count of distinct curves was given in the previous chapter and is given by q. So taking into account the redundancy, $\Phi(A \otimes B)$ consists of the q curves R_{l_1,l_2} with local systems of rank $q\frac{d}{q}k_1k_2 = dk_1k_2$.

Let us compute the relevant local system $(\mathcal{S}_1 \boxtimes \mathcal{S}_2) \circ \mathcal{O}$. Because the curves in $(L_1 \times L_2) \times_{E \times E} \mathcal{L}$ are in homotopy classes $(\frac{s}{d}, \frac{r}{d}, ...)$ in $L_1 \times L_2 \times \mathcal{L}$, under pullback to the intersection we get the local system $V_1 \otimes V_2$ with monodromy $M_1^{\frac{s}{d}} \otimes M_2^{\frac{r}{d}}$. Pushing this forward by a d-th cover gives the rank dk_1k_2 local system we expected.

Case 2

Suppose

$$A = \pi_{r*}(L \otimes F(V_1, \exp(N_1))) \tag{7.30}$$

and

$$B = S(x, V_2, N_2) \tag{7.31}$$

where V's and N's are as before, and x is the point where B is supported. The tensor product in the derived category is equivalent to the object

$$A \otimes B = \pi_{r*}(L \otimes F(V_1, \exp(N_1))) \otimes S(x, V_2, N_2) \tag{7.32}$$

because A is locally free. Since B is supported at a point, $A \otimes B$ is supported at the same point as well. Since A is locally isomorphic to $\mathcal{O}_{E_\tau}^{\oplus k}$, where k is the dimension of the vector bundle A, we get that $A \otimes B$ is isomorphic to $B^{\oplus k}$. Its image is thus $\Phi(B)^{\oplus k}$.

On the other hand the Lagrangian defining $\Phi(A)\hat{\otimes}\Phi(B)$, given by $(L_1 \times L_2) \circ \mathcal{L}$ is exactly the supporting Lagrangian of B. The local system is the local system of B to the power $r \cdot k_1$, which is exactly the dimension of the vector bundle A. This establishes that again $\Phi(A)\hat{\otimes}\Phi(B) = \Phi(A \otimes B)$.

Case 3

Suppose

$$A = S(x, V_1, N_1) \tag{7.33}$$

and

$$B = S(y, V_2, N_2) \tag{7.34}$$

Since we are working on a curve we can find a locally free resolution of A of length 2 of the form

$$0 \to \mathcal{R}_2 \to \mathcal{R}_1 \to 0 \tag{7.35}$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are of the same dimension since they are isomorphic away from x. The tensor product is

$$0 \to \mathcal{R}_2 \otimes B \to \mathcal{R}_1 \otimes B \to 0 \tag{7.36}$$

The sheaves in the above sequence are supported at y and if $x \neq y$ the above map is an isomorphism and so this object of the derived category is equivalent to 0. If $x = y$ let us use an isomorphism to identify A with $\mathcal{O}_{k_1 x}$ and B with $\mathcal{O}_{k_2 x}$ where k_i is the rank of V_i. Using now the standard 2-term resolution for O_{kx} we find that the derived tensor product is $\mathcal{O}_{k_i x}[-1] \oplus \mathcal{O}_{k_i x}$ where k_i is the smaller of the two.

On the Lagrangian side, if $x = a_1\tau + b_1$ and $y = a_2\tau + b_2$ satisfy $a_1 \neq a_2$ we get that $\Phi(A)\hat{\otimes}\Phi(B)$ is 0 because the fibers underlying $\Phi(A)$ and $\Phi(B)$ are disjoint, which is the desired outcome. In the case $a_1 = a_2$ but $b_1 \neq b_2$ we get the case of $L_1 \times L_2$ and $\mathcal{L}$ forming a non-transverse sequence, corresponding to taking $\hat{\otimes}$ of a fiber with itself with local system of different eigenvalues, and thus the solution is again 0. In case $x = y$ the proof of Theorem 6.1 shows that $\Phi(A)\hat{\otimes}\Phi(B)$ is the sum $F[-1] \oplus F$ (where F is the vertical through $-a$) equipped with whichever of $\exp(2\pi i b_i 1_{V_i} + N_i)$ is of smaller rank, which is the mirror of the above two term complex.

Now we can say that Φ is a monoidal functor, and similarly for its inverse that gives the mirror symmetry equivalence. We conclude that Φ is in fact a monoidal equivalence for $(D^{alg}, \otimes)$ and $(D^{symp}, \hat{\otimes})$.

□

Bibliography

[1] M.F. Atiyah. Vector bundles over an elliptic curve. *Proceedings of the London Mathematical Society*, pages 414–452, 1957.

[2] D. Auroux. Special lagrangian fibrations, wall-crossing, and mirror symmetry. *Surveys in Differential Geometry*, **13**:1–47, 2009.

[3] P. Balmer. Presheaves of triangulated categories and reconstruction of schemes. *Mathematische Annalen*, **324**:558–580, 2002.

[4] P. Balmer. The spectrum of prime ideals in tensor triangulated categories. *Journal fur die Reine und Angewandte Mathematik*, **588**:149–168, 2005.

[5] J.J. Duistermaat. On global action-angle coordinates. *Communications on Pure and Applied Mathematics*, **33**:687–706, 2006.

[6] M. Kashiwara and P. Shapira. *Sheaves on Manifolds.* Springer, 1990.

[7] M. Kontsevich. Homological algebra of mirror symmetry. *Proceedings of the International Congress of Mathematicians*, 1994.

[8] N.C. Leung, S.T. Yau, and E. Zaslow. From special lagrangians to hermitian-yang-mills via fourier-mukai transform. *Proceedings of the Winter School on Mirror Symmetry*, **23**, 2001.

[9] K.G. Max. *Basic Concepts of Enriched Category Theory.* Cambridge University Press, 1982.

[10] D. Mumford. *Abelian varieties.* Oxford University Press, 1986.

[11] T. Oda. Vector bundles on an elliptic curve. *Nagoya Math. J.*, **43**:41–72, 1971.

[12] A. Polishchuk and E. Zaslow. Categorical mirror symmetry: the elliptic curve case. *Proceedings of the Winter School on Mirror symmetry, Vector Bundles and Lagrangian Submanifolds*, **23**, 2001.

[13] A. Strominger, S.T. Yau, and E. Zaslow. Mirror symmetry is t-duality. *Proceedings of the Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds*, **23**, 2001.

[14] K. Wehrheim and C. Woodward. Floer cohomology and geometric composition of lagrangian correspondences. *arXiv:0905.1368*, 2007.

[15] K. Wehrheim and C. Woodward. Functoriality for lagrangian correspondences in floer theory. *arXiv:0708.2851*, 2007.

[16] K. Wehrheim and C. Woodward. Pseudoholomorphic quilts. *arXiv:0905.1369*, 2007.

[17] K. Wehrheim and C. Woodward. Quilted floer cohomology. *arXiv:0905.13708*, 2007.