A NEW APPROACH TO Symp

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Abstract. In this note, we show that a linear $(A_\infty, 2)$-algebra A as in [BC] induces an A_∞-structure on the bar complex $TA[1]$.

For $r \geq 1$ and $\mathbf{n} \in \mathbb{Z}^r_{\geq 0} \setminus \{\mathbf{0}\}$ there is a 2-associahedron $W_\mathbf{n}$. We would like to associate to $W_\mathbf{n}$ an operation which takes in $|\mathbf{n}|$ Floer cochains and returns a Floer cochain, and we can do so. The issue is when we try to write down the relations satisfied by these operations. The codimension-1 strata of $W_\mathbf{n}$ are fiber products of lower-dimensional $W_\mathbf{m}$'s, so we cannot write down A_∞-like relations amongst these operations. This forces us to make an end-run, e.g. by defining the 2-composition operations in Symp to be cobordisms between spaces rather than morphisms of complexes.

The purpose of this note is to describe an alternate setup: instead of an operation for each $r \geq 1, \mathbf{n} \in \mathbb{Z}^r_{\geq 0} \setminus \{\mathbf{0}\}$, I propose defining an operation $C_{(\mathbf{n}^i)}$ for each $r \geq 1, a \geq 0, \mathbf{n}^1, \ldots, \mathbf{n}^a \in \mathbb{Z}^r_{\geq 0} \setminus \{\mathbf{0}\}$. The underlying domain moduli space is the fiber product $W_{\mathbf{n}^1} \times_{K_r} \cdots \times_{K_r} W_{\mathbf{n}^a}$. This operation therefore takes in $|\mathbf{n}^1| + \cdots + |\mathbf{n}^a|$ inputs and returns a outputs. The reward for this more extensive collection of operations is that there are A_∞-like relations amongst them, since the strata of $W_{\mathbf{n}^1} \times_{K_r} \cdots \times_{K_r} W_{\mathbf{n}^a}$ are products of fiber products of 2-associahedra. Moreover, this seemingly-complicated setup becomes simpler when we phrase it in a way analogous to how an A_∞-algebra structure can be seen as a differential-graded co-algebra structure.

We will make things easier for ourselves by ignoring signs.

0.1. Presentation of an A_∞-algebra as a differential-graded co-algebra.

Let $\mathcal{A}$ be an A_∞-algebra. This means that it is a graded vector space with operations

$$\mu^s \colon \mathcal{A}^{\otimes s} \to \mathcal{A}[2-s], \quad s \geq 0, \tag{1}$$

where $\mathcal{A}[2-s]$ is $\mathcal{A}$ with the grading shifted down by $2-s$. (Note in particular that μ^1 decreases the grading by 1.)

Now consider the tensor co-algebra $T\mathcal{A}[1] := \bigoplus_{i \geq 0} \mathcal{A}[1]^{\otimes i}$. We can define a map $\widehat{\mu} \colon T\mathcal{A}[1] \to T\mathcal{A}[1]$ by

$$\widehat{\mu}(x_1 \otimes \cdots \otimes x_k) := \sum_{i,j} x_1 \otimes \cdots \otimes x_i \otimes \mu^j(x_{i+1} \otimes \cdots \otimes x_{i+j}) \otimes x_{i+j+1} \otimes \cdots \otimes x_k. \tag{2}$$

The A_∞-equations satisfied by μ imply $\widehat{\mu}^2 = 0$. Moreover, for each summand in $\widehat{\mu}(x_1 \otimes \cdots \otimes x_k)$, the number of terms is decreased by $j-1$, while the application of μ^j decreases the grading by $2-j$, so overall the grading is decreased by $(2-j)+(j-1) = 1$. That is, $\widehat{\mu}$ decreases the grading by 1, so we can think of it as a differential.

There is probably of way of doing this "in reverse", but for now we content ourselves with the fact that we can make an A_∞-structure on $\mathcal{A}$ into a differential on $T\mathcal{A}[1]$.

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0.2. **An A_∞-structure on $T\mathcal{A}[1]$.** For $r \geq 1$, $a \geq 0$, and $\mathbf{m}^1, \ldots, \mathbf{m}^a \in \mathbb{Z}^r_{\geq 0} \setminus \{\mathbf{0}\}$, write $\mathbf{n} \coloneqq \mathbf{m}^1 + \cdots + \mathbf{m}^a$ and define an operation

$$C_{(\mathbf{m}^i)} \colon \mathcal{A}^{\otimes n_1} \otimes \cdots \otimes \mathcal{A}^{\otimes n_r} \to \mathcal{A}^{\otimes a}\big[2 + a - |\mathbf{n}| - r\big]. \tag{3}$$

Think of this as the result of counting curves with domains in the fiber product $W_{\mathbf{m}^1} \times_{K_r} \cdots \times_{K_r} W_{\mathbf{m}^a}$.

We will now use these operations to define a flat A_∞-structure on $T\mathcal{A}[1]$. For $r \geq 1$, define an operation $\widehat{C}_r \colon T\mathcal{A}[1]^{\otimes r} \to T\mathcal{A}[1]$ like so:

$$\widehat{C}_1 \begin{pmatrix} x_{11} \\ \vdots \\ x_{1n_1} \end{pmatrix} \coloneqq \sum_{i,j} \left(\begin{array}{c} x_{11} \\ \vdots \\ x_{1i} \\ C_{((j))} \begin{pmatrix} x_{1(i+1)} \\ \vdots \\ x_{1(i+j)} \end{pmatrix} \\ x_{1(i+j+1)} \\ \vdots \\ x_{1n_1} \end{array} \right), \tag{4}$$

$$\widehat{C}_r \begin{pmatrix} x_{11} & & x_{r1} \\ \vdots & \cdots & \vdots \\ x_{1n_1} & & x_{rn_r} \end{pmatrix} \coloneqq \sum_{\substack{i \leq j \\ \mathbf{m}^1 + \cdots + \mathbf{m}^a = \mathbf{n}}} C_{(\mathbf{m}^k)} \begin{pmatrix} x_{(i+1)1} & & x_{(i+j)1} \\ \vdots & \cdots & \vdots \\ x_{(i+1)n_{i+1}} & & x_{(i+j)n_{i+j}} \end{pmatrix}, \qquad r \geq 2.$$

Let's calculate what $\widehat{C}_r$ does to degrees. We consider the $r = 1$ and $r \geq 2$ cases separately.

- $\widehat{C}_1$ reduces the number of factors of $\mathcal{A}[1]$ by $j - 1$, while the application of $C_{((j))}$ reduces the degree by $2 - j$. The overall effect is therefore to reduce the degree by $(2 - j) + (j - 1) = 1$.
- For $r \geq 2$, $\widehat{C}_r$ reduces the number of factors by $|\mathbf{n}| - a$, while the application of $C_{(\mathbf{m}^k)}$ reduces the degree by $2 + a - |\mathbf{n}| - r$. The overall effect is therefore to reduce the degree by

$$(2 + a - |\mathbf{n}| - r) + (|\mathbf{n}| - a) = 2 - r. \tag{5}$$

For $r \geq 1$, $\widehat{C}_r$ therefore reduces degree by $2 - r$.

References

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