Pseudoholomorphic Quilts with Figure Eight Singularity

by

Nathaniel Sandsmark Bottman

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Author .... Signature redacted ....
Department of Mathematics
August 1, 2015

Certified by.. Signature redacted ....
Katrin Wehrheim
Associate Professor, UC Berkeley
Thesis Supervisor

Accepted by .. Signature redacted ....
Alexei Borodin
Chairman, Department Committee on Graduate Theses

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Abstract

In this thesis, I prove several results toward constructing a machine that turns Lagrangian correspondences into A_∞-functors between Fukaya categories. The core of this construction is pseudoholomorphic quilts with figure eight singularity.

In the first part, I propose a blueprint for constructing an algebraic object that binds together the Fukaya categories of many different symplectic manifolds. I call this object the "symplectic A_∞-2-category Symp". The key to defining the structure maps of Symp is the figure eight bubble.

In the second part, I establish a collection of strip-width–independent elliptic estimates. The key is function spaces which augment the Sobolev norm with another term, so that the norm of a product can be bounded by the product of the norms in a manner which is independent of the strip-width. Next, I prove a removable singularity theorem for the figure eight singularity. Using the Gromov compactness theorem mentioned in the following paragraph, I adapt an argument of Abbas–Hofer to uniformly bound the norm of the gradient of the maps in cylindrical coordinates centered at the singularity. I conclude by proving a "quilted" isoperimetric inequality.

In the third part, which is joint with Katrin Wehrheim, I use my collection of estimates to prove a Gromov compactness theorem for quilts with a strip of (possibly non-constant) width shrinking to zero. This features local C^∞-convergence away from the points where energy concentrates. At such points, we produce a nonconstant quilted sphere.

Thesis Supervisor: Katrin Wehrheim
Title: Associate Professor, UC Berkeley

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This thesis is dedicated to my family.

Contents

Chapter 1

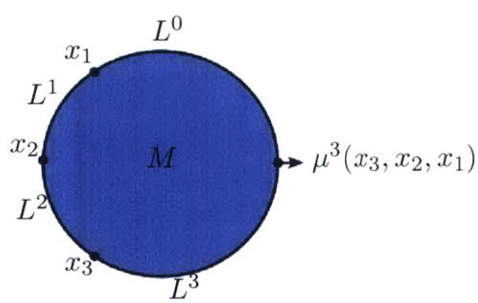
Introduction: A blueprint for Symp

In [Fl], Floer proved the Arnol'd conjecture for a Lagrangian L in M assuming $\pi_2(M, L) = 0$, a hypothesis which guarantees that "disk bubbling" is not a concern. Fukaya's innovation was to embrace disks: in [Fu], he introduced the **Fukaya $\mathbf{A}_\infty$-category**, whose objects are Lagrangians submanifolds (with some additional structures) and where the morphisms from L^0 to L^1 are formal sums of points in $L^0 \cap L^1$ in the case that these Lagrangians have transverse intersection. The structure maps

$$\mu^d : \hom(L^{d-1}, L^d) \otimes \cdots \otimes \hom(L^0, L^1) \to \hom(L^0, L^d)$$

are defined by counting pseudoholomorphic maps from disks with d input and 1 output boundary marked points to M and with boundary conditions in the L_i's, as on the left of Figure 1-1.

Figure eight bubbling arose in [WeWo1] as a conjectural obstruction to an identity of quilted Floer homology groups. In this thesis (parts of which are published in [Bo1], [BoWe2]), I follow Fukaya's example of embracing singularity formation and describe a program to relate immersed Fukaya categories of different symplectic manifolds by counting figure eight bubbles with seam marked points, one of which is pictured on the right of Figure 1-1. The main goal of this program is to construct an **$\mathbf{A}_\infty$ 2-category Symp**, whose objects are compact symplectic manifolds and where the 1-morphisms from M_0 to M_1 are given by $\mathrm{Fuk}(M_0^- \times M_1)$, the immersed Fukaya A_∞ category of $(M_0 \times M_1, (-\omega_{M_0}) \oplus \omega_{M_1})$.



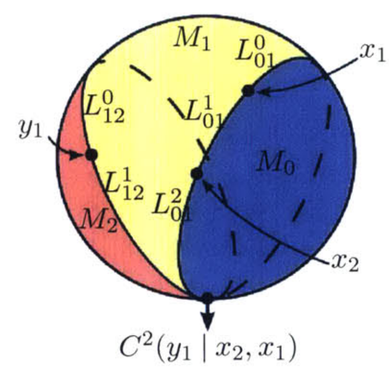


Figure 1-1: On the left is one of the domains for the maps whose count defines μ^3, one of the A_∞ structure maps in $\mathrm{Fuk}(M)$. On the right is a domain for a 2-patch eight with marked points, whose count will define $C^2(y_1 \mid x_2, x_1)$.

Symp will fulfill the goal of Wehrheim–Woodward's **quilted Floer theory**, which aims to introduce functorial methods to the study of the Fukaya A_∞ category. §1.1 summarizes the results proven in this thesis, which are proven in §2 and §3. In §1.2, I describe the proposed structure maps of Symp and the relations they should satisfy; I note the places where the results of §2 and §3 will be needed. In §1.3, I explain how Symp should unite several existing constructions, and should yield new relations amongst them.

1.1 Results in §2 and §3

The novel feature of the figure eight bubble is the "singularity" where the seams intersect tangentially, and one of the main results of my thesis is a Removal of Singularity Theorem when $L_{01} \circ L_{12}$ is cleanly immersed. The other main result is a Gromov Compactness Theorem for **strip-shrinking**, a prototype of which is the degeneration where the two seams of a figure eight bubble come together and fuse. Both results will be crucial for constructing meaningful moduli spaces of figure eights.

1.1.1 Removal of singularity for figure eight bubbles

We say that Lagrangians $L_{01} \subset M_0^- \times M_1$ and $L_{12} \subset M_1^- \times M_2$ have **cleanly immersed composition** if the intersection

$$L_{01} \times_{M_1} L_{12} = (L_{01} \times L_{12}) \cap (M_0 \times \Delta_{M_1} \times M_2)$$

is transverse (which implies that the "composition" $L_{01} \circ L_{12} := \pi_{02}(L_{01} \times_{M_1} L_{12})$ is an immersed Lagrangian in $M_0^- \times M_2$) and furthermore any two local branches of $L_{01} \circ L_{12}$ meet cleanly.

A figure eight bubble between L_{01} and L_{12} is a tuple of finite energy pseudoholomorphic maps

$$w_0 : \mathbb{R} \times (-\infty, -\tfrac{1}{2}] \to M_0, \qquad w_1 : \mathbb{R} \times [-\tfrac{1}{2}, \tfrac{1}{2}] \to M_1, \qquad w_2 : \mathbb{R} \times [\tfrac{1}{2}, \infty) \to M_2$$

satisfying the seam conditions

$$(w_0(s, -\tfrac{1}{2}), w_1(s, -\tfrac{1}{2})) \in L_{01}, \quad (w_1(s, \tfrac{1}{2}), w_2(s, \tfrac{1}{2})) \in L_{12} \qquad \forall s \in \mathbb{R}.$$

By stereographic projection, the total domain of a figure eight bubble can be transformed to a punctured sphere, similar to the illustration in Figure 1-1. In §2 I establish the following basic analytic property of figure eight bubbles, which was conjectured in [WeWo4].

Removal of Singularity Theorem 2.1.2: *If the composition $L_{01} \circ L_{12}$ is cleanly immersed, then w_0 resp. w_2 extend to continuous maps on $D^2 \cong (\mathbb{R} \times (-\infty, -\frac{1}{2}]) \cup \{\infty\}$ resp. $D^2 \cong (\mathbb{R} \times [\frac{1}{2}, \infty)) \cup \{\infty\}$, and $w_1(s, -)$ converges to constant paths as $s \to \pm\infty$.*

The proof has two parts. First, I show that in cylindrical coordinates centered at the singularity, the gradients are uniformly bounded. This goes by contradiction: if not, I bubble off a nonconstant quilted sphere using Gromov Compactness Theorem 3.3.1. Second, I establish an isoperimetric inequality for the energy in the quilted setting, which crucially relies on the cleanness hypothesis. Since the argument is local at the singularity, the theorem applies also to figure eight bubbles with marked points.

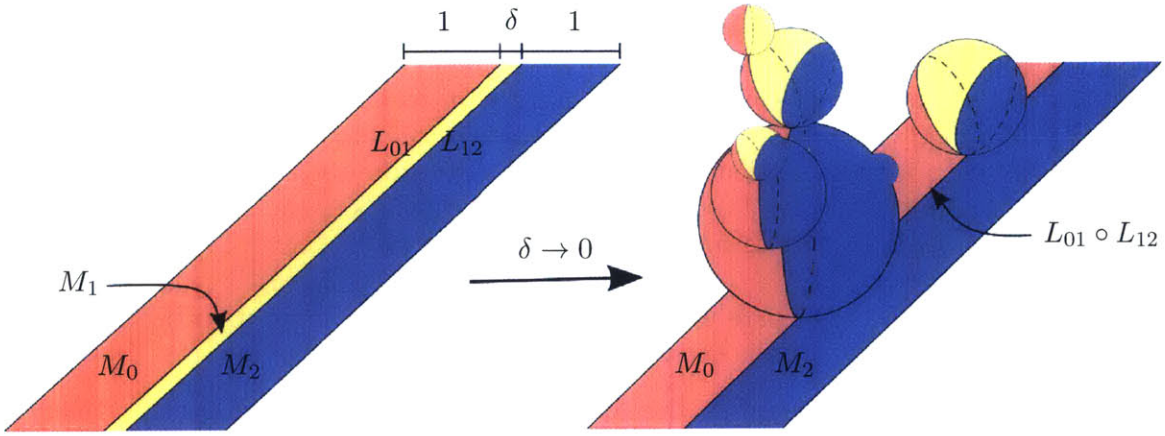


Figure 1-2: The strip-shrinking degeneration and the tree of quilted spheres that results.

1.1.2 Gromov compactness for strip-shrinking

A phenomenon that is new to quilted Floer theory is strip-shrinking [WeWo1], in which the width of a strip or annulus in a quilted surface shrinks to zero. To understand the topology of moduli spaces of maps from such domains, one would like a "Gromov Compactness Theorem": if the **energy** (i.e. the summed L^2-norms of the derivatives) is bounded, then a subsequence of the maps converges $\mathcal{C}^\infty_{\rm loc}$ away from finitely many points where the gradient blows up, and at each blowup point a tree of quilted spheres forms; Figure 1-2 illustrates this degeneration. Wehrheim and Woodward [WeWo1] established compactness up to energy concentration in the special case of embedded composition of $L_{01} \circ L_{12}$ (which in particular requires the composed Lagrangian to be embedded, not immersed), though only in an $H^2 \cap W^{1,4}$-topology and with a lower bound on the energy concentration that has no geometric interpretation but arises by contradiction from mean value inequalities. In joint work with Katrin Wehrheim [BoWe2], we establish full $\mathcal{C}^\infty_{\rm loc}$-convergence in the most general natural case. Our current proof produces a single quilted sphere — rather than a whole tree of spheres — at each blowup point, hence may not capture all energy, but I have recently developed an argument to fix this shortcoming. The results in this chapter will be stated for "immersed composition of $L_{01} \circ L_{12}$", which is slightly weaker than cleanly-immersed composition.

A key for the proof is to establish "energy quantization" for figure eight bubbles, which are the one new type of bubble that can form at the blowup points.

Lower Energy Bound Lemma 3.2.8: For fixed almost complex structures and Lagrangians with immersed composition $L_{01} \circ L_{12}$, the energy of nontrivial figure eight bubbles is bounded below by a positive quantity.

Energy quantization plus the elliptic estimates discussed in the next subsection allow us to establish the following theorem.

Gromov Compactness Theorem 3.3.1: Let $\underline{Q}^\nu$ be a sequence of quilted surfaces containing an annulus or strip Q_1^ν of widths $\delta^\nu \to 0$. Label the patches of $\underline{Q}^\nu$ with a fixed tuple $\underline{M}$ of compact symplectic manifolds, let M_1 and M_0, M_2 be the labels of Q_1^ν and the adjacent patches, and fix compatible almost complex structures over each patch. Fix compact Lagrangian seam conditions for each seam of $\underline{Q}^\nu$ so that the Lagrangian correspondences L_{01}, L_{12} associated to the seams of Q_1^ν have immersed composition $L_{01} \circ L_{12}$. Now suppose that $(\underline{v}^\nu)_{\nu\in\mathbb{N}} : \underline{Q}^\nu \to \underline{M}$ is a sequence of pseudoholomorphic quilts of bounded energy with the given Lagrangian seam conditions.

Then there exists a subsequence that converges up to bubbling to a punctured quilt $\underline{v}^\infty : \underline{Q}^\infty \setminus Z \to (\underline{M} \setminus M_1)$. Here $\underline{Q}^\infty$ is the quilted surface obtained as limit of the $\underline{Q}^\nu$ by replacing Q_1^ν with a seam labeled by $L_{01} \circ L_{12}$, Z is a finite set of bubbling points, $\underline{v}^\infty$ satisfies seam conditions in the fixed Lagrangian correspondences and for the new seam in $L_{01} \circ L_{12}$, and convergence holds in the following sense:

- The energy densities $|\mathrm{d}\underline{v}^\nu|^2$ are uniformly bounded on every compact subset of $\underline{Q}^\infty \setminus Z$, and at each point in Z there is energy concentration of at least $\hbar > 0$;
- The quilt maps $\underline{v}^\nu|_{\underline{Q}^\nu \setminus (Q_1^\nu \cup Z)}$ on the complement of Z in the patches other than Q_1^ν converge with all derivatives on every compact set to $\underline{v}^\infty$. If $L_{01} \circ L_{12}$ is cleanly immersed, then $\underline{v}^\nu$ extends continuously to Z. If moreover $L_{01} \circ L_{12} \subset M_0 \times M_2$ is a smooth submanifold, then this extension is smooth.
- At least one type of bubble forms at each point $z \in Z$ in the following sense: there is a sequence of (tuples of) maps obtained by rescaling the maps defined on the various patches near z, which converges $\mathcal{C}^\infty_{\mathrm{loc}}$ to one of the following:
  - a nonconstant, finite-energy pseudoholomorphic map $\mathbb{R}^2 \to M_\ell$ to one of the symplectic manifolds in $\underline{M}$ (this can be completed to a nonconstant pseudoholomorphic sphere in M_ℓ);
  - a nonconstant, finite-energy pseudoholomorphic map $\mathbb{H} \to M_\ell^- \times M_k$ to a product of symplectic manifolds associated to the patches on either side of a seam in $\underline{Q}^\nu$, that satisfies the corresponding Lagrangian seam condition (this can be extended to a nonconstant pseudoholomorphic disk in $M_\ell^- \times M_k$, in particular including the cases of disks with boundary on $L_{01} \subset M_0^- \times M_1$ or $L_{12} \subset M_1^- \times M_2$);
  - a nonconstant, finite-energy map $\mathbb{H} \to M_0^- \times M_2$ with boundary condition in $L_{01} \times L_{12}$ with lift to $L_{01} \times_{M_1} L_{12}$ (the singularity can be removed in the case of cleanly-immersed composition; the same holds for the next type of bubble);
  - a nonconstant, finite-energy figure eight bubble.

1.1.3 Width-independent elliptic estimates

The analytic core of the two results just described is a substantial strengthening of the strip-shrinking estimates in [WeWo1] — in particular, from embedded to immersed geometric composition. In §2 I construct a special connection that allows me to obtain estimates without boundary terms for quilted Cauchy–Riemann operators, with uniform constants for all small widths of a strip. This allows us to strengthen the uniform $H^2 \cap W^{1,4}$ estimates of [WeWo1] to H^{k+1} and thus $\mathcal{C}^{k-1}$ for any $k \geq 1$, which is e.g. needed to deduce nontriviality of bubbles with generalized boundary condition in $L_{01} \circ L_{12}$.

The estimate I prove for any $k \geq 0$ is

$$\|\zeta\|_{\widetilde{H}^{k+1}(Q_{r,\delta})} \leq C\big(\|\mathcal{D}\zeta\|_{\widetilde{H}^k(Q_{R,\delta})} + \|\zeta\|_{H^0(Q_{R,\delta})}\big),$$

where $Q_{\rho,\delta}$ is a quilt with total domain $(-\rho,\rho)^2$ and seams at $(-\rho,\rho) \times \{\pm\delta\}$, ζ is a section of the tangent bundles of M_0, M_1, M_2 by a triple of maps u_0, u_1, u_2 from the bottom resp.

middle resp. top patches to M_0 resp. M_1 resp. M_2, $\mathcal{D}$ is the linearized Cauchy–Riemann operator, and $\widetilde{H}^k$ is a modification of the Sobolev space H^k:

$$\begin{aligned}\|\zeta(s,t)\|_{\widetilde{H}^k} &= \|\zeta(s,t)\|_{H^k} + \sum_{l=0}^{k-2} \sup_{t_0\in(-\rho,\rho)} \|\nabla^l\zeta(s,t_0)\|_{H^1(s\in(-\rho,\rho))} \\ &=: \|\zeta(s,t)\|_{H^k} + \sum_{l=0}^{k-2} \|\nabla^l\zeta(s,t)\|_{C^0_t H^1_s}\end{aligned}$$

For fixed ρ and δ, the H^k- and $\widetilde{H}^k$-norms are equivalent due to the embedding $H^1 \hookrightarrow C^0$ for 1-dimensional domains. However, this equivalence is not uniform in δ. The utility of the $\widetilde{H}^k$-norm is that the $C^0_t H^1_s$-terms satisfy a product rule that is uniform in δ, since the H^1-norm is applied on the domain $(-\rho,\rho)$.

1.2 Construction of the A_∞ 2-category Symp

Throughout this section, μ^k_{ij} will denote the k-ary composition operation in the A_∞ category $\mathrm{Fuk}(M_i^- \times M_j)$.

1.2.1 The A_∞ bifunctor Symp

We begin our blueprint of Symp by modifying the figure eight bubble by placing $l_2 \geq 0$ input marked points on the 12-seam (between the M_1 patch and the M_2 patch), $l_1 \geq 0$ input marked points on the 01-seam, and one output marked point at the singular point of the quilt (the left half of Figure 1-1 is the $l_2 = 1, l_1 = 2$ case). The 12-seam is now divided into l_2+1 segments; label these by Lagrangians $L^0_{12}, \ldots, L^{l_2}_{12} \subset M_1^- \times M_2$. Label the segments of the 01-seam by $L^0_{01}, \ldots, L^{l_1}_{01} \subset M_0^- \times M_1$. Given a finite-energy pseudoholomorphic quilt with this domain, and assuming that $L^0_{01} \circ L^0_{12}$ and $L^{l_2}_{01} \circ L^{l_1}_{12}$ are cleanly immersed and that the branches of one intersect the branches of the other cleanly, it follows[1] from Removal of Singularity Theorem 2.1.2 that the limit of the three maps at the output marked point is a generator of $CF^*(L^0_{01} \circ L^0_{12}, L^{l_2}_{01} \circ L^{l_1}_{12})$. The 0-dimensional moduli space of such quilts should therefore define a map

$$\begin{aligned} C^2(- \mid -) :\ & CF^*(L^{l_2-1}_{12}, L^{l_2}_{12}) \otimes \cdots \otimes CF^*(L^0_{12}, L^0_{12}) \\ & \otimes CF^*(L^{l_1-1}_{01}, L^{l_1}_{01}) \otimes \cdots \otimes CF^*(L^0_{01}, L^1_{01}) \\ & \longrightarrow CF^*(L^0_{01} \circ L^0_{12}, L^{l_1}_{01} \circ L^{l_2}_{12}). \end{aligned} \tag{1.1}$$

The first step toward combining these quilts into a moduli space will be building a Deligne–Mumford-like compactification of their domains, as a manifold with boundary with corners. For that purpose note that the moduli spaces of quilted disks constructed in [MaWo] to represent Stasheff's multiplihedra can be viewed as configuration space of marked points on one of the two seams of the domain of figure eight bubbles. Allowing for marked points on both seams is analogous to the construction of "biassociahedra" in [MaWeWo], and will again yield families of marked quilted surfaces parametrized by singular polyhedra. We can

[1] Removal of Singularity Theorem 2.1.2 is stated for figure eight bubbles with no input marked points (i.e. with seams mapping to single Lagrangians L_{01}, L_{12}). However, the proof is local at the singularity, so it also applies to figure eights with marked points.

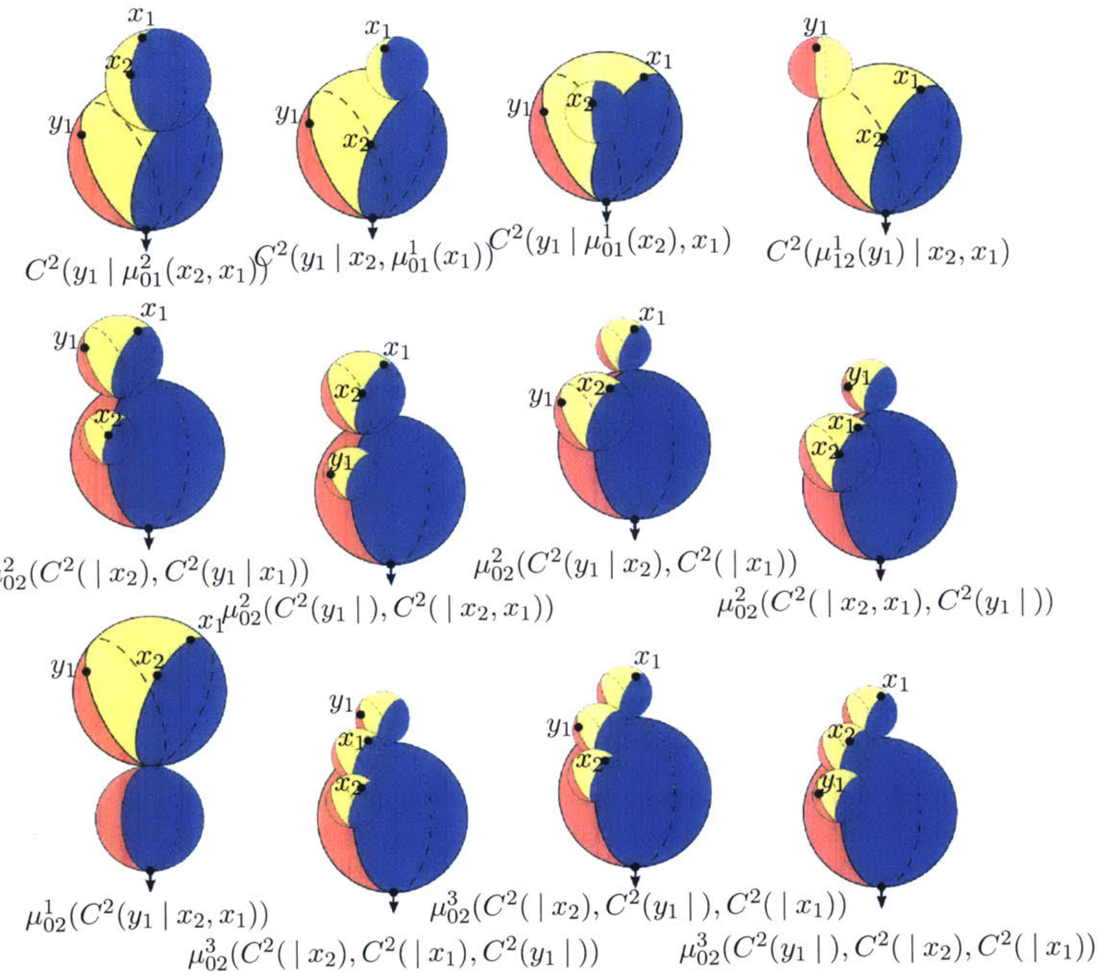


Figure 1-3: These quilted surfaces represent on the one hand the algebraic expressions in the bifunctor relation of Conjecture 1.2.1 (with the exception of curvature terms), and on the other hand the expected boundary strata of the 1-dimensional moduli space of figure eight bubbles with one marked point on the 01-seam and two on the 12-seam (with the exception of bubbling that does not involve marked points).

then build moduli spaces of pseudoholomorphic quilts whose domains are given by points in a biassociahedron.

The boundaries of 1-dimensional moduli spaces of such quilts will then give rise to a collection of relations among these maps. These boundary components will arise from several effects. Firstly, the underlying biassociahedron of quilted surfaces has boundary. In the example of Figure 1-3, its top strata correspond to the eight algebraic terms not involving μ^1-terms. Secondly, Floer trajectories can break off at each marked point on a seam. In the example of Figure 1-3, this corresponds to the three algebraic terms in the first row involving pre-composition with μ^1_{01} or μ^1_{12}. Moreover, energy concentrating at the outgoing marked point (where in cylindrical coordinates two pairs of 01- and 12-seams approach each other asymptotically) can be captured geometrically as a Floer trajectory for the composed Lagrangians breaking off. In the example of Figure 1-3, this corresponds to the algebraic term in the bottom left corner involving post-composition with μ^1_{02}. Together, these algebraic terms capture the relations describing an A_∞ bifunctor. Finally, energy can concentrate without marked points being involved, yielding sphere, disk, or figure eight bubbles. Spheres will be interior points of the ambient polyfold, hence do not contribute to the algebraic relation. Disk bubbling can appear on a 01- or 12-seam, yielding algebraic terms involving pre-composition with μ_{01}^0 or μ_{12}^0, which reflect curvature of an A_∞ algebra associated to a Lagrangian L_{01}^i or L_{12}^i. Figure eight bubbling can only appear when the underlying quilted surface also approaches a boundary face of the biassociahedron that can be interpreted as the width of the middle strip shrinking to zero. In the example of Figure 1-3 these are the configurations in the second and third row corresponding to post-composition with μ_{02}^k for $k \geq 1$. Since figure eight bubbling does not add to the corner index, we expect additional boundary faces arising from adding any number of figure eight bubbles without marked points to the 02-seams of these configurations. Algebraically, this will be reflected by $C()$ terms in any number of entries of μ_{02}^k, meaning that the A_∞ bifunctor itself is curved. More precisely, we expect for each $l_2, l_1 \geq 0$ a relation of the form

$$\begin{aligned}
&\sum C(y_{l_2}, \dots, y_{k+m+1}, \mu_{12}^l(y_{k+m}, \dots, y_{k+1}), y_k, \dots, y_1 \mid x_{l_1}, \dots, x_1) \; + \\
&\quad + \; \sum C(y_{l_2}, \dots, y_1 \mid x_{l_1}, \dots, x_{k+m+1}, \mu_{01}^l(x_{k+m}, \dots, x_{k+1}), x_k, \dots, x_1) = \\
&\qquad = \sum \mu_{02}^n(C(y_{l_2}, \dots, y_{l_2-k_n+1} \mid x_e, \dots, x_{l_1-m_n+1}), \dots, C(y_{k_1}, \dots, y_1 \mid x_{m_1}, \dots, x_1)).
\end{aligned}$$

For instance, the twelve summands not involving curvature terms in the $l_2 = 1, l_1 = 2$ case, together with their corresponding boundary strata, are shown in Figure 1-3. These relations are exactly what is required of a curved A_∞ bifunctor, so we are led to the following conjecture.

Conjecture 1.2.1. *Given compact symplectic manifolds M_0, M_1, M_2, there is a curved A_∞ bifunctor*

$$C^2\colon (\mathrm{Fuk}(M_1^- \times M_2), \mathrm{Fuk}(M_0^- \times M_1)) \to \mathrm{Fuk}(M_0^- \times M_2)$$

between Fukaya categories of cleanly-immersed Lagrangians that sends a pair of Lagrangians (L_{12}, L_{01}) with cleanly-immersed composition to $L_{01} \circ L_{12}$ and is defined on the morphism level by the maps $C^2(- \mid -)$ in (1.1).

Remark 1.2.2 (Immersed Fukaya categories). Conjecture 1.2.1 is naturally stated in the immersed setting: Even if the source Fukaya categories were chosen to contain only embedded Lagrangians as objects, the target Fukaya category would still need to contain immersed Lagrangians, since Hamiltonian perturbation of Lagrangian correspondences L_{01}, L_{12} can always achieve cleanly-immersed but generally not embedded composition. The Fukaya categories in Conjecture 1.2.1 will have as objects immersed Lagrangians $\varphi\colon L \to M$ with clean self-intersections; boundary conditions in L for a map $u\colon \Sigma \to M$ on $\gamma \subset \partial\Sigma$ will require the data of a continuous lift of $u|_\gamma$ to L. Cleanly-immersed Lagrangian boundary conditions have been discussed in various settings before, but [BoWel] will develop both analysis and algebra from scratch — the first since we choose to work in the framework of polyfolds, and the second since our version of the immersed Fukaya category requires control of sheet-switching at self-intersection points in terms of cochain labels which encode contributions from sheet switching figure eight bubbles.

Remark 1.2.3. Special cases of Conjecture 1.2.1 will yield A_∞-functors similar to the ones constructed in [MaWeWo]. The main differences are that [MaWeWo] works with *extended*

Fukaya categories (whose objects are composable sequences of embedded Lagrangian correspondences) and is necessarily limited to settings (such as monotonicity) in which figure eight bubbling is excluded.

(i) For $M_2 = \mathrm{pt}$ the restriction of C to a fixed unobstructed object $L_{01} \in \mathrm{Fuk}(M_0^- \times M_1)$ yields a curved A_∞-functor $C_{L_{01}}\colon \mathrm{Fuk}(M_1^-) \to \mathrm{Fuk}(M_0^-)$. On the object level, this functor sends $L_1 \subset M_1^-$ to $L_{01} \circ L_1 \subset M_0^-$ if this composition is cleanly immersed; on the morphism level, this functor is defined by the maps $C(-\,|\,)$ arising from the moduli spaces of quilted disks with a nonnegative number of marked points on the boundary circle, as on the right of Figure 1-1.

(ii) For $M_0 = \mathrm{pt}$ the restriction of C to a fixed unobstructed object $L_{12} \in \mathrm{Fuk}(M_1^- \times M_2)$ yields a curved A_∞-functor ${}_{L_{12}}C\colon \mathrm{Fuk}(M_1) \to \mathrm{Fuk}(M_2)$ that sends $L_1 \subset M_1$ to $L_1 \circ L_{12} \subset M_2$ if this composition is cleanly immersed, and is defined by the maps $C(\,|\,-)$ on morphism level.

(iii) We expect the special case C_{Δ_M} resp. ${}_{\Delta_M}C$ of both functors to be the identity functor on $\mathrm{Fuk}(M \times \mathrm{pt}) \cong \mathrm{Fuk}(M) \cong \mathrm{Fuk}(\mathrm{pt}^- \times M)$. More generally, we will show in [BoWe1] that $(C_{L_{12}^T}, {}_{L_{12}}C)$ and $({}_{L_{12}}C, C_{L_{12}^T})$ form adjoint pairs, where $L_{12}^T \subset M_2^- \times M_1$ is obtained from $L_{12} \subset M_1^- \times M_2$ by exchange of factors.

1.2.2 The k-ary operations C^k

The construction described in the previous subsection can be extended to quilted spheres with $k+1$ patches, where the seams are k circles that all intersect tangentially at the south pole, and where we allow a nonnegative number of input marked points on each seam and regard the south pole as the output marked point. (The figure eight bubble is the $k = 2$ case.) Considering 0-dimensional moduli spaces should define maps

$$\begin{aligned} C^k(- \mid \cdots \mid -) :\ & CF^*(L_{(k-1)k}^{l_k-1}, L_{(k-1)k}^{l_k}) \otimes \cdots \otimes CF^*(L_{(k-1)k}^{0}, L_{(k-1)}^{0}) \otimes \cdots \\ & \cdots \otimes CF^*(L_{01}^{l_1-1}, L_{01}^{l_1}) \otimes \cdots \otimes CF^*(L_{01}^{0}, L_{01}^{1}) \\ & \longrightarrow CF^*(L_{01}^0 \circ \cdots \circ L_{(k-1)k}^0, L_{01}^{l_1} \circ \cdots \circ L_{(k-1)k}^{l_k}). \end{aligned} \tag{1.2}$$

Note that since quilted spheres with two patches can be identified by disks mapping to the product, C^1 will be equal to the map sending k cochains $(x_k, \ldots, x_1)$ to the A_∞ composition $\mu_{01}^k(x_k, \ldots, x_1)$.

Considering the boundary strata of 1-dimensional moduli spaces of such quilts, along with the gluing analysis in [BoWe2], leads us to the following conjecture, which generalizes Conjecture 1.2.1.

Conjecture 1.2.4. *For $k \geq 1$, the C^k's satisfy a relation (Rk), beginning with:*

$$\sum C^1(\cdots, C^1(\cdots), \cdots) = 0 \tag{R1}$$

$$\begin{aligned} \sum C^2(\cdots, C^1(\cdots), \cdots \mid \cdots) + \sum C^2(\cdots \mid \cdots, C^1(\cdots), \cdots) + \\ + \sum C^1(C^2(\cdots \mid \cdots), \ldots, C^2(\cdots \mid \cdots)) = 0. \end{aligned} \tag{R2}$$

We therefore plan to define an $\mathbf{A_\infty}$ **2-category** to be a collection of objects, 1-morphisms, 2-morphisms, and k-ary composition maps C^k that send a k-tuple of composable 1-morphisms

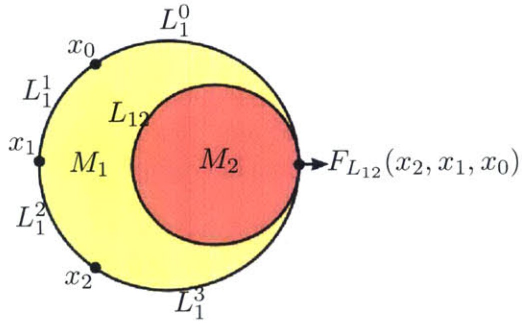


Figure 1-4: One of the domains for the ternary part of $F_{L_{12}}$. Setting $M_0 := \text{pt}$ in Figure 1-1 is equivalent to deleting the blue patch, which leaves us with a quilted disk.

to a single 1-morphism, send an $(l_k, \ldots, l_1)$-tuple of 2-morphisms to a single 2-morphism as in (1.2), and satisfy the relations (Rk) in Conjecture 1.2.4.

1.3 Applications of **Symp** and relations with existing constructions

1.3.1 C^2 specializes to "symplectic Fourier–Mukai transforms"

When we set $M_0 = \text{pt}$ and restrict C^2 to a fixed object $L_{12} \in \text{Fuk}(M_1 \times M_2)$, we obtain a curved A_∞ functor $F_{L_{12}} : \text{Fuk}(M_1) \to \text{Fuk}(M_2)$. On the object level, this functor sends $L_1 \subset M_1$ to $L_1 \circ L_{12} \subset M_2$, and on the morphism level, this functor is defined by the maps $C^2(\,|\,-)$ arising from the moduli spaces of quilted disks with a nonnegative number of marked points on the boundary circle, as in the right half of Figure 1-1.

These A_∞ functors are similar to those defined in [MaWeWo] between extended Fukaya categories. Mau–Wehrheim–Woodward sidestepped serious analytical difficulties by straightening the seams in a neighborhood of the output marked point, at the price of producing a less geometric functor. Furthermore, there is a formal similarity between our A_∞ functors and **Fourier–Mukai transforms**, functors between derived categories of coherent sheaves that are a standard tool in algebraic geometry. Kontsevich's Homological Mirror Symmetry Conjecture predicts that the Fukaya category is dual to the derived category of coherent sheaves, so we expect our A_∞ functors — "symplectic Fourier–Mukai transforms" — to be dual in some precise sense to Fourier–Mukai transforms via mirror symmetry.

"Composition commutes with categorification"

Relation (R3) of Conjecture 1.2.4 has an immediate consequence for symplectic Fourier–Mukai transforms:

Corollary of Conjecture 1.2.4. *Assume Conjecture 1.2.4. If M_1, M_2, M_3 are compact symplectic manifolds and L_{12} and L_{23} are objects of* $\text{Fuk}(M_1^- \times M_2)$ *and* $\text{Fuk}(M_2^- \times M_3)$*, then the curved A_∞ functors $C_{L_{12} \circ L_{23}}$ and $C_{L_{23}} \circ C_{L_{12}}$ are homotopic.*

Proof. In (R3), set $M_0 := \text{pt}$ and fix $L_{12} \subset M_1^- \times M_2$ and $L_{23} \subset M_2^- \times M_3$. □

This is the analogue of the "categorification commutes with composition" statements made for the analogous A_∞ functors between extended Fukaya categories in [MaWeWo, WeWo2].

Adjunction properties implied by the relations in Symp

In this subsubsection we derive another consequence of Conjecture 1.2.4: transposing a Lagrangian correspondence gives rise to adjoint pairs of symplectic Fourier–Mukai transforms. The impetus for considering adjunction properties came from the following fact about Fourier–Mukai transforms.

Proposition 1.3.1 (Prop. 5.9, [Hu]). *For any object $\mathcal{P}_{12} \in \mathrm{D^bCoh}(X_1 \times X_2)$, set*

$$\mathcal{P}_{12,L} := \mathcal{P}_{12}^\vee \otimes \pi_2^* \omega_{X_2}[\dim(X_2)], \qquad \mathcal{P}_{12,R} := \mathcal{P}_{12}^\vee \otimes \pi_1^* \omega_{X_1}[\dim(X_1)],$$

where ω_{X_i} is the canonical bundle of X_i. Then the Fourier–Mukai transforms

$$G_{\mathcal{P}_{12,L}} : \mathrm{D^bCoh}(X_2) \to \mathrm{D^bCoh}(X_1), \qquad G_{\mathcal{P}_{12,R}} : \mathrm{D^bCoh}(X_2) \to \mathrm{D^bCoh}(X_1)$$

are left resp. right adjoint to $G_{\mathcal{P}_{12}}$.

The analogous property for symplectic Fourier–Mukai transforms depends on Conjecture 1.2.4 and the following, much smaller, conjecture.

Conjecture 1.3.2. *Given a compact symplectic manifold M, the following triangle commutes:*

$$\begin{tikzcd}
(\mathrm{Fuk}(M^-), \mathrm{Fuk}(M)) \arrow[r, "C"] \arrow[dr, "\Delta"'] & \mathrm{Fuk}(\mathrm{pt}) \arrow[d, equal] \\
& \mathbb{Z}/2\mathbb{Z}\text{-Vect}\,.
\end{tikzcd}$$

Here Δ is the diagonal bimodule, where $\mathrm{Fuk}(M^-)$ *has been identified with* $\mathrm{Fuk}(M)^{\mathrm{op}}$.

Corollary 2 of Conjectures 1.2.4 and 1.3.2. *Assume Conjectures 1.2.4 and 1.3.2. If M_1, M_2 are compact symplectic manifolds and L_{12} is an object of* $\mathrm{Fuk}(M_1^- \times M_2)$, *then $(C_{L_{12}}, C_{L_{12}^T})$ is an adjoint pair.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
(\mathrm{Fuk}(M_1), \mathrm{Fuk}(M_1^- \times M_2), \mathrm{Fuk}(M_2^-)) \arrow[r, "{(C^2,\mathrm{id})}"] \arrow[d, "{(\mathrm{id},C^2)}"'] & (\mathrm{Fuk}(M_2), \mathrm{Fuk}(M_2^-)) \arrow[d, "C^2"] \arrow[dr, "\Delta"] & \\
(\mathrm{Fuk}(M_1), \mathrm{Fuk}(M_1^-)) \arrow[r, "C^2"] \arrow[rr, bend right, "\Delta"'] & \mathrm{Fuk}(\mathrm{pt}) \arrow[r, equal] & \mathbb{Z}/2\mathbb{Z}\text{-Vect}\,.
\end{tikzcd}$$

By Conjecture 1.2.4, the inner square commutes; by Conjecture 1.3.2, the triangles commute. Given $L_{12} \in \mathrm{Fuk}(M_1^- \times M_2)$, the commutativity of the outer square implies that $(F_{L_{12}}, F_{L_{12}^T})$ is an adjoint pair. $\square$

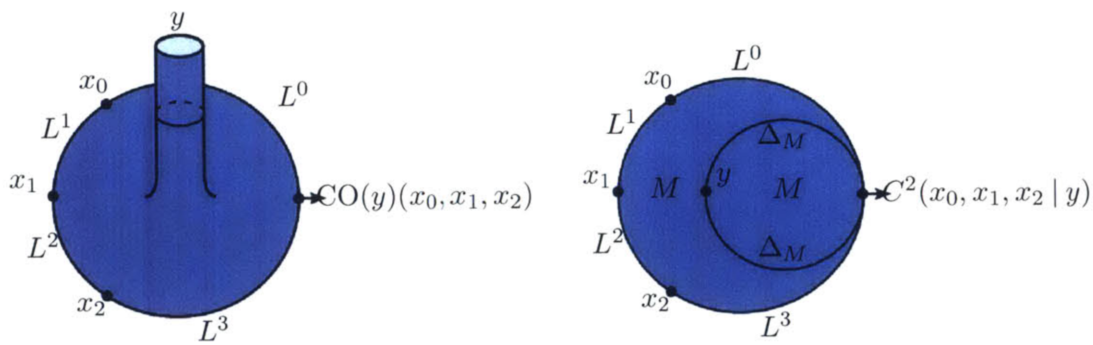


Figure 1-5: The two moduli spaces whose identification would yield Conjecture 1.3.3.

1.3.2 The closed-open string map as a specialization of C^2

The **closed-open string map** [Abo] is a homomorphism $\mathrm{CO} : QC^*(M) \to CC^*(\mathrm{Fuk}(M))$ from quantum cochains of a closed symplectic manifold to Hochschild cochains over its A_∞ category. It is an important tool for studying deformations of $\mathrm{Fuk}(M)$, and is a crucial ingredient in Abouzaid's generation criterion for $\mathrm{Fuk}(M)$ (which has so far only been proven in the open, wrapped setting). In this subsection we conjecture that the closed-open string map is a specialization of geometric composition. This point of view was (at least implicitly) suggested in [Ga] and [RiSm].

Conjecture 1.3.3. *After identifying $QC^*(M)$ with $CF^*(\Delta_M, \Delta_M)$ and $CC^*(\mathrm{Fuk}(M))$ with $\hom(\mathrm{id}_{\mathrm{Fuk}(M)}, \mathrm{id}_{\mathrm{Fuk}(M)})$, the closed-open string map $\mathrm{CO} : QC^*(M) \to CC^*(\mathrm{Fuk}(M))$ agrees with the geometric composition bifunctor $C^2 : (\mathrm{Fuk}(M), \mathrm{Fuk}(M^- \times M)) \to \mathrm{Fuk}(M)$ when applied to a nonnegative number of morphisms in $\mathrm{Fuk}(M)$ and one endomorphism of Δ_M.*

Figure 1-5 illustrates the two moduli spaces whose identification would confirm Conjecture 1.3.3.

Corollary 3 of Conjectures 1.2.4 and 1.3.3. *Assume Conjectures 1.2.4 and 1.3.3. Then the following diagram commutes:*

$$\begin{tikzcd}
QH^*(M_1) \arrow[r] \arrow[d, "H(\mathrm{CO})"'] & HF^*(L_{12}, L_{12}) \arrow[d] & QH^*(M_2) \arrow[l] \arrow[d, "H(\mathrm{CO})"] \\
HH^*(\mathrm{Fuk}(M_1)) \arrow[r] & HH^*(\mathrm{Fuk}(M_1), F^*_{L_{12}} \mathrm{Fuk}(M_2)) & HH^*(\mathrm{Fuk}(M_2)). \arrow[l]
\end{tikzcd}$$

Here the upper horizontal arrows are induced by the geometric composition bifunctor, after identifying quantum cohomology with Floer cohomology of the diagonal. The middle vertical arrow uses the identification $CC^*(\mathcal{A}, F^*\mathcal{B}) = \hom(F, F)$ for $F : \mathcal{A} \to \mathcal{B}$ an A_∞ functor. The bottom arrows come from the contra- resp. covariant functoriality of Hochschild cohomology in the first resp. second variables.

Proof. The commutativity of the left resp. right squares follows from Conjecture 1.2.4 applied to $(\mathrm{Fuk}(M_1), CF^*(\Delta_{M_1}, \Delta_{M_1}), L_{12})$ resp. $(\mathrm{Fuk}(M_1), L_{12}, CF^*(\Delta_{M_2}, \Delta_{M_2}))$. □

I expect that in situations where Hochschild homology and cohomology are dual (see [Ga]), partially dualizing the outer square in this conjectural corollary will yield a square similar to [RiSm, Theorem 1.3].

Chapter 2

Removing the figure eight singularity

In the case of **embedded composition** of L_{01} and L_{12}, where the projection

$$\pi_{02} : L_{01} \times_{M_1} L_{12} \to M_0^- \times M_2$$

is injective and hence a Lagrangian embedding, monotonicity and Maslov index assumptions allowed Wehrheim–Woodward [WeWo1] to establish an isomorphism of quilted Floer cohomologies (as defined in [WeWo4])

$$HF(\ldots, L_{01}, L_{12}, \ldots) \cong HF(\ldots, L_{01} \circ L_{12}, \ldots). \tag{2.1}$$

The analytic core of the proof was a **strip-shrinking degeneration**, in which a triple of pseudoholomorphic strips coupled by Lagrangian seam conditions degenerates to a pair of strips, via the width of the middle strip shrinking to zero. The monotonicity and embeddedness assumptions allowed for an implicit exclusion of all bubbling, which was conjectured to include a novel **figure eight bubbling** that (unlike disk or sphere bubbling) could be an algebraic obstruction to (2.1).

Gromov Compactness Theorem 3.3.1 proves that a blowup of the gradient in a sequence of pseudoholomorphic quilts with an annulus or strip of shrinking width gives rise to one of the standard bubbling phenomena (pseudoholomorphic spheres and disks) or a nontrivial **figure eight bubble**, as depicted in Figure 2-1. In this chapter we apply this Gromov

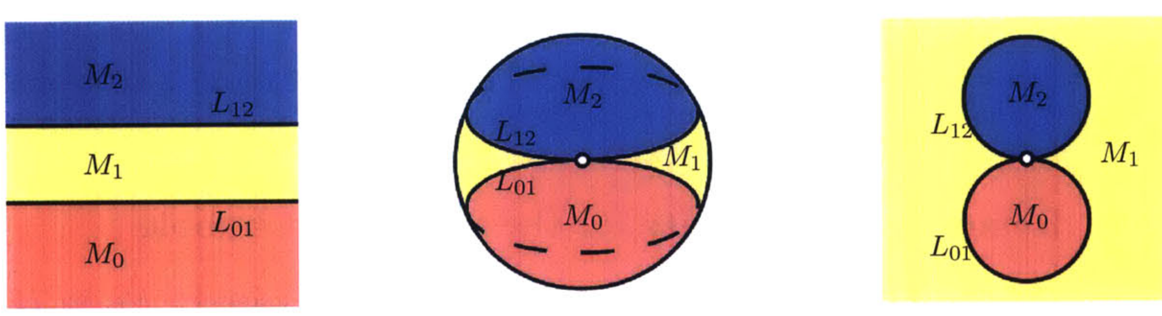


Figure 2-1: The left figure illustrates a figure eight bubble, the middle figure illustrates its reparametrization as a pseudoholomorphic quilt whose domain is the punctured sphere, and the right figure illustrates an inverted figure eight (defined in §2.1, and equivalent to the left figure via $z \mapsto -1/z$). The domain of the left and right figures is $\mathbb{C}$, and the point at infinity in the left figure corresponds to the punctures in the middle and right figures.

Compactness Theorem to show that the figure eight singularity can be removed, as [WeWo1] conjectured:

Removal of Singularity Theorem 2.1.2: If the composition $L_{01} \circ L_{12}$ is **cleanly immersed** (immersed, and in addition the local branches of $L_{01} \circ L_{12}$ intersect one another cleanly), then w_0 resp. w_2 extend to continuous maps on $D^2 \cong (\mathbb{R} \times (-\infty, 0]) \cup \{\infty\}$ resp. $D^2 \cong (\mathbb{R} \times [0, \infty)) \cup \{\infty\}$, and $w_1(s, -)$ converges to constant paths as $s \to \pm\infty$. If $L_{01} \circ L_{12}$ is embedded, then the latter limits are equal.

This theorem is the first step in the program outlined in §1, which proposes a collection of composition operations amongst Fukaya categories of different symplectic manifolds.

In support of §1, Appendix A also proves the analogous removal of singularities for pseudoholomorphic disks with a type of immersed boundary values in $L_{01} \circ L_{12}$, under the assumption that the latter is cleanly-immersed resp. immersed. These results are not necessarily new, see Appendix A for citations, but provided for the sake of completeness. It is also conceptually useful to recast the (possibly singular) disk bubbles with boundary on $L_{01} \circ L_{12}$ as **squashed eight bubbles**, that is as triples of finite energy pseudoholomorphic maps

$$w_0 \colon \mathbb{R} \times (-\infty, 0] \to M_0, \qquad w_1 \colon \mathbb{R} \to M_1, \qquad w_2 \colon \mathbb{R} \times [0, \infty) \to M_2$$

satisfying the generalized seam condition

$$\big(w_0(s,0), w_1(s), w_1(s), w_2(s,0)\big) \in L_{01} \times_{M_1} L_{12} \qquad \forall\, s \in \mathbb{R}.$$

In §2.2 we establish a collection of strip-width–independent elliptic estimates that allow for nonstandard domain complex structure. This is necessitated by the following analytic formulation for the figure eight singularity: In cylindrical coordinates for a neighborhood of infinity, the two seams become two pairs of curves approaching each other asymptotically (see the right figure in Figure 2-1). On finite cylinders, the standard complex structure on this quilted surface can be pulled back to a quilted surface in which the width of the strips is constant and the complex structures are nonstandard, but converge in $\mathcal{C}^0$ and stay within a controlled $\mathcal{C}^k$-distance from the standard structure for any $k \geq 1$.

The hypothesis that M_0, M_1, M_2 are closed is not essential: As explained in §3, it is enough for the symplectic manifolds to be geometrically bounded and to have a priori $\mathcal{C}^0$-bounds on the various pseudoholomorphic curves. In a future paper we will treat the noncompact setting in a more systematic way.

2.1 Removal of singularity for the figure eight bubble

In this section and the next we will be working with symplectic manifolds M_0, M_1, M_2, almost complex structures J_0, J_1, J_2, and pseudoholomorphic curves with seam conditions defined by compact Lagrangian correspondences

$$L_{01} \subset M_0^- \times M_1, \qquad L_{12} \subset M_1^- \times M_2, \tag{2.2}$$

with $L_{01} \circ L_{12}$ either immersed or cleanly immersed:

- L_{01} and L_{12} have **immersed composition** if the intersection

$$L_{01} \times_{M_1} L_{12} = (L_{01} \times L_{12}) \cap (M_0 \times \Delta_{M_1} \times M_2)$$

  is transverse. This implies that $\pi_{02}\colon L_{01} \times_{M_1} L_{12} \to M_0^- \times M_2$ is a Lagrangian immersion, e.g. by [WeWo4, Lemma 2.0.5], and in this situation we will denote the image by $L_{01} \circ L_{12} := \pi_{02}(L_{01} \times_{M_1} L_{12})$.

- If L_{01}, L_{12} have immersed composition and furthermore any two local branches of $L_{01} \circ L_{12}$ intersect cleanly — i.e. at any intersection of two local branches there is a chart for $M_0^- \times M_2$ (as a smooth manifold) in which each of those two branches is identified with an open subset of a vector subspace of $\mathbb{R}^n$ — then the composition $L_{01} \circ L_{12}$ is **cleanly immersed**.

The purpose of §2.1 is to prove a removal of singularity theorem for inverted figure eight bubbles.

Definition 2.1.1. An **inverted figure eight bubble between $\mathbf{L_{01}}$ and $\mathbf{L_{12}}$** is a triple of smooth maps

$$\underline{w} = \begin{pmatrix} w_0\colon \overline{B}_1(-i) \setminus \{0\} \to M_0 \\ w_1\colon \mathbb{C}^* \setminus (B_1(i) \cup B_1(-i)) \to M_1 \\ w_2\colon \overline{B}_1(i) \setminus \{0\} \to M_2 \end{pmatrix}$$

satisfying the Cauchy–Riemann equations $\partial_s w_\ell + J_\ell(w_\ell)\partial_t w_\ell = 0$ for $\ell \in \{0,1,2\}$ and the seam conditions

$$(w_0(-i+e^{i\theta}), w_1(-i+e^{i\theta})) \in L_{01} \quad \forall\, \theta \neq \tfrac{\pi}{2}, \qquad (w_1(i+e^{i\theta}), w_2(i+e^{i\theta})) \in L_{12} \quad \forall\, \theta \neq \tfrac{3\pi}{2},$$

and which have finite energy

$$\textstyle\int w_0^*\omega_0 + \int w_1^*\omega_1 + \int w_2^*\omega_2 \;=\; \frac{1}{2}\Big(\int |\mathrm{d}w_0|^2 + \int |\mathrm{d}w_1|^2 + \int |\mathrm{d}w_2|^2\Big) < \infty,$$

where we have endowed M_ℓ with the metric

$$g_\ell := \omega_\ell(-, J_\ell -). \tag{2.3}$$

Throughout §2.1, the norm of a tangent vector on M_ℓ will always be defined using g_ℓ.

Fix for §2.1 closed symplectic manifolds M_0, M_1, M_2, compatible almost complex structures $J_\ell \in \mathcal{J}(M_\ell, \omega_\ell)$, $\ell \in \{0,1,2\}$, compact Lagrangians L_{01}, L_{12} as in (2.2) with cleanly-immersed composition, and an inverted figure eight bubble $\underline{w}$ between L_{01} and L_{12}.

In fact, only the arguments in §2.2 require the composition $L_{01} \circ L_{12}$ to be cleanly immersed, rather than just immersed, but we assume the stronger hypothesis throughout §2.1 for cohesiveness.

The following theorem says that the singularity at 0 of a figure eight bubble can be continuously removed, under the hypothesis of cleanly-immersed composition.

Theorem 2.1.2. *The maps w_0, w_2 continuously extend to 0, and the limits $\lim_{z\to 0,\,\mathrm{Re}(z)>0} w_1(z)$ and $\lim_{z\to 0,\,\mathrm{Re}(z)<0} w_1(z)$ both exist. If moreover the immersion $\pi_{02}\colon L_{01}\times_{M_1} L_{12} \to M_0^- \times M_2$ is an embedding, then the latter limits are equal so that w_1 also extends continuously to* 0.

The proof of this theorem draws on the removal of singularity strategies in [AbbHo, §7.3] and in [McSa, §4.5]. First, we follow [AbbHo] and establish a uniform gradient bound in cylindrical coordinates near the puncture (Lemma 2.1.4), which we use to show that the lengths of the paths $\theta \mapsto w_\ell(\epsilon e^{i\theta})$ converge to zero as $\epsilon \to 0$ (Lemma 2.1.3). The substantial modification to the argument of [AbbHo] is that we must use the Gromov Compactness Theorem 2.1.2 in order to prove uniform gradient bounds in Lemma 2.1.4. Once we have proven that lengths go to zero, we follow [McSa] and prove an isoperimetric inequality for the energy (Lemma 2.1.8), which we use to show that the energy on disks around the puncture decays exponentially with respect to the logarithm of the radius. Here the nontrivial modification is in the quilted nature of our isoperimetric inequality. Finally, an argument from [AbbHo] allows us to conclude that w_0 and w_2 extend continuously to the puncture. The continuous extension of w_1 follows from the gradient bound in cylindrical coordinates and the immersed composition of L_{01} and L_{12}. The formal proof of Theorem 2.1.2 is given in §2.1.2.

2.1.1 Lengths tend to zero

The first step toward the Removal of Singularity Theorem 2.1.2 is to show that the lengths of the paths $\theta \mapsto w_\ell(\epsilon e^{i\theta})$ converge to zero as $\epsilon \to 0$. This is nontrivial since the conformal structure of the quilted surface near the singularity does not allow us to apply mean value inequalities effectively, as in previous removal of singularity results for pseudoholomorphic curves. Hence the finiteness of energy only provides a sequence $\epsilon^\nu \to 0$ along which the lengths tend to zero. This allowed Bottman–Wehrheim to deduce a weak removal of singularity in §3, but the stronger Theorem 2.1.2 will require the full strength of the generalized strip-shrinking analysis developed in §2.2 and the resulting Gromov Compactness Theorem 3.3.1.

In this subsection we will work in cylindrical coordinates centered at the singularity, hence we define the reparametrized maps

$$v_\ell(s,t) := w_\ell\big(e^{2\pi(s+it)}\big) \qquad \text{for} \quad \ell \in \{0,1,2\}, \tag{2.4}$$

whose domains $V_0, V_1, V_2 \subset (-\infty, 0] \times \mathbb{R}/\mathbb{Z}$ are given by

$$V_0 := \big\{(s,t) \;\big|\; s \le 0,\ |t - \tfrac{3}{4}| \le \tfrac{1}{4} - \theta(s)\big\}, \qquad V_2 := \big\{(s,t) \;\big|\; s \le 0,\ |t - \tfrac{1}{4}| \le \tfrac{1}{4} - \theta(s)\big\},$$
$$V_1 := \big\{(s,t) \;\big|\; s \le 0,\ |t - \tfrac{1}{2}| \le \theta(s) \ \vee\ |t - 1| \le \theta(s)\big\},$$

with

$$\theta(s) := \tfrac{1}{2\pi}\arcsin\big(\tfrac{1}{2}e^{2\pi s}\big). \tag{2.5}$$

Now the paths $w_\ell(\epsilon e^{i\theta})$ for fixed $\epsilon \in (0,1]$ correspond to the following paths for fixed $s = \frac{\log \epsilon}{2\pi} \le 0$:

$$\gamma_s^0 := v_0(s,-) \;:\; [\tfrac{1}{2} + \theta(s), 1 - \theta(s)] \longrightarrow M_0, \qquad \gamma_s^2 := v_2(s,-) \;:\; [\theta(s), \tfrac{1}{2} - \theta(s)] \longrightarrow M_2,$$
$$\gamma_s^1 := v_1(s,-) \;:\; [\tfrac{1}{2} - \theta(s), \tfrac{1}{2} + \theta(s)] \cup [1 - \theta(s), 1 + \theta(s)] \longrightarrow M_1. \tag{2.6}$$

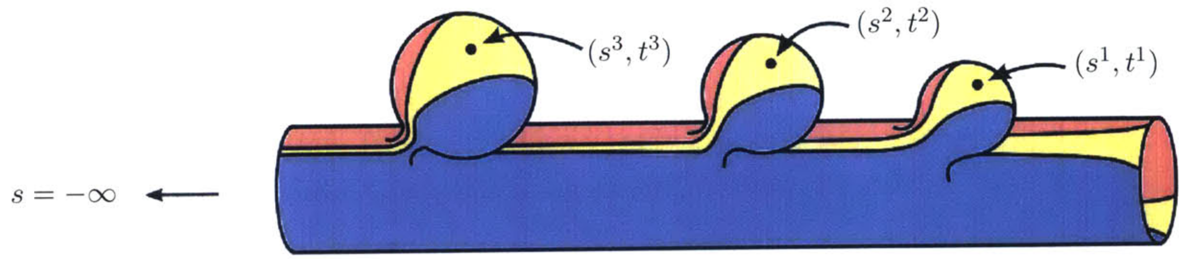


Figure 2-2: To prove Lemma 2.1.4, we assume that the cylindrical reparametrizations v_ℓ do not have uniformly bounded gradient, then bubble off a nonconstant quilted map. In this illustration, the bubbled-off map is a figure eight bubble.

The length of γ_s^ℓ is given by the integral $\ell(\gamma_s^\ell) := \int |\frac{\mathrm{d}}{\mathrm{d}t}\gamma_s^\ell| \,\mathrm{d}t$ over the respective domain, and will be controlled by the following main result of this subsection.

Lemma 2.1.3. *The L^2-lengths of the paths $\gamma_s^0, \gamma_s^1, \gamma_s^2$ defined in* (2.6) *converge to zero as $s \to -\infty$:*

$$\int_{1/2+\theta(s)}^{1-\theta(s)} |\tfrac{\mathrm{d}}{\mathrm{d}t}\gamma_s^0|^2 \,\mathrm{d}t + \left(\int_{1/2-\theta(s)}^{1/2+\theta(s)} + \int_{1-\theta(s)}^{1+\theta(s)}\right) |\tfrac{\mathrm{d}}{\mathrm{d}t}\gamma_s^1|^2 \,\mathrm{d}t + \int_{\theta(s)}^{1/2-\theta(s)} |\tfrac{\mathrm{d}}{\mathrm{d}t}\gamma_s^2|^2 \,\mathrm{d}t \underset{s\to-\infty}{\longrightarrow} 0.$$

In particular, the length $\ell(\gamma_s) := \ell(\gamma_s^0) + \ell(\gamma_s^1) + \ell(\gamma_s^2)$ tends to zero as $s \to -\infty$.

The proof of Lemma 2.1.3 will use ideas from [AbbHo]. The novel difficulty — due to the conformal structure — is to establish the following uniform gradient bound on $|\mathrm{d}\underline{v}|$, the upper semicontinuous function defined by

$$|\mathrm{d}\underline{v}|\colon (-\infty,0] \times \mathbb{R}/\mathbb{Z} \to [0,\infty), \qquad |\mathrm{d}\underline{v}(s,t)|^2 := |\mathrm{d}v_0(s,t)|^2 + |\mathrm{d}v_1(s,t)|^2 + |\mathrm{d}v_2(s,t)|^2, \tag{2.7}$$

where the functions $|\mathrm{d}v_\ell(s,t)|$ are set to zero where they are not defined.

Lemma 2.1.4. *The gradient $|\mathrm{d}\underline{v}|$ defined in* (2.7) *is uniformly bounded.*

We prove Lemma 2.1.4 by contradiction: if $|\mathrm{d}v_\ell|$ is not bounded for some ℓ, then there is a sequence of points (s^ν, t^ν) (necessarily with $s^\nu \to -\infty$) at which $|\mathrm{d}v_\ell|$ diverges. Rescaling at these points produces a nonconstant quilted map, as illustrated in Figure 2-2, but this contradicts the finite-energy hypothesis on $\underline{v}$. The technical input the Gromov Compactness Theorem 3.3.1. This theorem is needed to deduce that the rescaled maps actually converge. In order to state it, we need to define the domains of the maps and a controlled fashion in which the strip-width can tend to zero.

The following definition is the only instance in §2.1 where we allow the almost complex structures to be domain-dependent, so that the notion of a squiggly strip quilt is flexible enough to be used in §2.2.

Definition 2.1.5. Fix $\rho > 0$, a real-analytic function $f\colon [-\rho,\rho] \to (0,\rho/2]$, domain-dependent compatible almost complex structures $J_\ell\colon [-\rho,\rho]^2 \to \mathcal{J}(M_\ell,\omega_\ell)$, $\ell \in \{0,1,2\}$, and a complex structure j on $[-\rho,\rho]^2$.

- A **$(\mathbf{J_0},\mathbf{J_1},\mathbf{J_2},\mathbf{j})$-holomorphic size-$(\mathbf{f},\rho)$ squiggly strip quilt for $(\mathbf{L_{01}},\mathbf{L_{12}})$** is a triple of smooth maps

$$\underline{v} = \begin{pmatrix} v_0\colon \{(s,t) \in (-\rho,\rho)^2 \mid t \le -f(s)\} \to M_0 \\ v_1\colon \{(s,t) \in (-\rho,\rho)^2 \mid |t| \le f(s)\} \to M_1 \\ v_2\colon \{(s,t) \in (-\rho,\rho)^2 \mid t \ge f(s)\} \to M_2 \end{pmatrix} \tag{2.8}$$

that fulfill the seam conditions

$$\big(v_0(s,-f(s)), v_1(s,-f(s))\big) \in L_{01}, \qquad \big(v_1(s,f(s)), v_2(s,f(s))\big) \in L_{12} \qquad \forall\, s \in (-\rho,\rho), \tag{2.9}$$

satisfy the Cauchy–Riemann equations

$$\mathrm{d}v_\ell(s,t) \circ j(s,t) - J_\ell(s,t,v_\ell(s,t)) \circ \mathrm{d}v_\ell(s,t) = 0 \qquad \forall\, \ell \in \{0,1,2\} \tag{2.10}$$

for (s,t) in the relevant domains, and have finite energy

$$E(\underline{v}) \;:=\; \textstyle\int v_0^*\omega_0 + \int v_1^*\omega_1 + \int v_2^*\omega_2 \;<\; \infty.$$

- A $(\mathbf{J_0}, \mathbf{J_2}, \mathbf{j})$**-holomorphic size-**$\rho$ **degenerate strip quilt for** $\mathbf{L_{01}} \times_{\mathbf{M_1}} \mathbf{L_{12}}$ **with singularities** is a triple of smooth maps

$$\underline{v} = \begin{pmatrix} v_0\colon\ (-\rho,\rho) \times (-\rho,0] \smallsetminus S \times \{0\} \to M_0 \\ v_1\colon\ (-\rho,\rho) \smallsetminus S \to M_1 \\ v_2\colon\ (-\rho,\rho) \times [0,\rho) \smallsetminus S \times \{0\} \to M_2 \end{pmatrix} \tag{2.11}$$

defined on the complement of a finite set $S \subset \mathbb{R}$ that fulfill the lifted seam condition

$$\big(v_0(s,0), v_1(s), v_1(s), v_2(s,0)\big) \in L_{01} \times_{M_1} L_{12} \qquad \forall\, s \in (-\rho,\rho) \smallsetminus S, \tag{2.12}$$

satisfy the Cauchy–Riemann equation (2.10) for $\ell \in \{0,2\}$ and (s,t) in the relevant domains, and have finite energy

$$E(\underline{v}) \;:=\; \textstyle\int v_0^*\omega_0 + \int v_2^*\omega_2 \;<\; \infty.$$

When j is the standard complex structure $i\colon \partial_s \mapsto \partial_t,\ \partial_t \mapsto -\partial_s$, the Cauchy–Riemann equation (2.10) can be expressed in coordinates as:

$$\partial_t v_\ell(s,t) - J_\ell(s,t,v_\ell(s,t))\partial_s v_\ell(s,t) = 0.$$

The novel hypothesis necessary for a sequence of squiggly strip quilts of widths $(f^\nu)_{\nu\in\mathbb{N}}$ to converge $\mathcal{C}^\infty_{\mathrm{loc}}$ away from the gradient blow-up points is that the widths "obediently shrink to zero":

Definition 2.1.6. Fix $\rho > 0$. A sequence $\left(f^\nu\right)_{\nu\in\mathbb{N}}$ of real-analytic functions $f^\nu\colon [-\rho,\rho] \to (0,\rho/2]$ **obediently shrinks to zero,** $\mathbf{f^\nu \Rightarrow 0}$, if $\max_{s\in[-\rho,\rho]} f^\nu(s) \underset{\nu\to\infty}{\longrightarrow} 0$ and

$$\sup_{\nu\in\mathbb{N}} \frac{\max_{s\in[-\rho,\rho]}\left|\frac{\mathrm{d}^k}{\mathrm{d}s^k} f^\nu(s)\right|}{\min_{s\in[-\rho,\rho]} f^\nu(s)} =: C_k < \infty \qquad \forall\, k \in \mathbb{N},$$

and in addition there are holomorphic extensions $F^\nu\colon [-\rho,\rho]^2 \to \mathbb{C}$ of $f^\nu(s) = F^\nu(s,0)$ such that (F^ν) converges $\mathcal{C}^\infty$ to zero.

We are finally in the position to bound the gradients of the reparametrized maps v_ℓ from (2.4).

Proof of Lemma 2.1.4. We will prove the equivalent statement that the "folded maps"

$$u_\ell\colon U_\ell \to M_\ell \times M_\ell^-, \qquad u_\ell(s,t) := (v_\ell(s,t), v_\ell(s, \tfrac{1}{2} - t)) \quad \text{for} \quad \ell = 0,1,2$$

have uniformly-bounded gradients, where the domains U_ℓ are given by

$$U_0 := \{(s,t) \mid s \leq 0,\ -\tfrac{1}{4} \leq t \leq -\theta(s)\}, \qquad U_2 := \{(s,t) \mid s \leq 0,\ \theta(s) \leq t \leq \tfrac{1}{4}\},$$
$$U_1 := \{(s,t) \mid s \leq 0,\ -\theta(s) \leq t \leq \theta(s)\}.$$

These maps are pseudoholomorphic with respect to the almost complex structures $\widehat{J}_\ell := J_\ell \oplus (-J_\ell)$ and satisfy the following boundary and seam conditions for $s \leq 0$:

$$u_0(s, -\tfrac{1}{4}) \in \Delta_{M_0}, \qquad (u_0(s,-\theta(s)), u_1(s,-\theta(s))) \in (L_{01} \times L_{01})^T,$$
$$u_2(s, \tfrac{1}{4}) \in \Delta_{M_2}, \qquad (u_1(s,\theta(s)), u_2(s,\theta(s))) \in (L_{12} \times L_{12})^T.$$

(Here $\theta(s) = \frac{1}{2\pi}\arcsin(\frac{1}{2}e^{2\pi s})$ as in (2.5), and $(L_{ij} \times L_{ij})^T$ is the image of $L_{ij} \times L_{ij}$ under the permutation $(x_i, x_j, y_i, y_j) \mapsto (x_i, y_i, x_j, y_j)$.) Finiteness of the energy of the inverted figure eight $\underline{w}$ translates into convergence of the integral $\lim_{S\to-\infty} \int_{(S,0]\times[-1/4,1/4]} \frac{1}{2}|\mathrm{d}\underline{u}|^2 < \infty$ of the energy density

$$|\mathrm{d}\underline{u}|\colon (-\infty,0] \times \left[-\tfrac{1}{4}, \tfrac{1}{4}\right] \to [0,\infty), \qquad |\mathrm{d}\underline{u}(s,t)|^2 := |\mathrm{d}u_0(s,t)|^2 + |\mathrm{d}u_1(s,t)|^2 + |\mathrm{d}u_2(s,t)|^2,$$

where the functions $|\mathrm{d}u_\ell(s,t)|$ are set to zero where they are not already defined (so $|\mathrm{d}\underline{u}|$ is upper semi-continuous). This convergence in particular implies

$$\int_{(-\infty,S]\times[-1/4,1/4]} \tfrac{1}{2}|\mathrm{d}\underline{u}|^2 \underset{S\to-\infty}{\longrightarrow} 0. \tag{2.13}$$

Now assume for a contradiction that there exists a sequence $(s^\nu, t^\nu) \in (-\infty,0]\times[-1/4,1/4]$ such that $|\mathrm{d}\underline{u}(s^\nu,t^\nu)| \to \infty$. Since the u_ℓ are smooth, this is possible only for $s^\nu \to -\infty$; passing to a further subsequence, we may in fact assume $s^{\nu+1} \leq s^\nu - 1$ and $s^1 \leq 1/4$. Depending on whether t^∞ is $\pm 1/4$ or is contained in $(-1/4,1/4)$, we derive a contradiction to (2.13):

$\mathbf{t^\infty = \pm 1/4}$. Assume $t^\infty = -1/4$; the $t^\infty = 1/4$ case can be treated in analogous fashion. Define a sequence (u_0^ν) by:

$$u_0^\nu\colon B_{1/8}(0) \cap \mathbb{H} \to M_0 \times M_0^-, \qquad u_0^\nu(s,t) := u_0(s + s^\nu, t - 1/4).$$

The map u_0^ν is $\widehat{J}_0$-holomorphic and satisfies the Lagrangian boundary condition $u_0(s,0) \in \Delta_{M_0}$ for $s \in (-1/8,1/8)$. Furthermore, $|\mathrm{d}u_0^\nu(0, t^\nu + 1/4)| \to \infty$, $t^\nu + 1/4 \to 0$ by assumption, and the energy of u_0^ν is bounded by the energy of $\underline{v}$, so [McSa, Lemma 4.6.5] implies the inequality $\liminf_{\nu\to\infty} \int_{B_{1/8}(0)} \frac{1}{2}|\mathrm{d}u_0^\nu|^2 > 0$, which contradicts (2.13).

$\mathbf{t^\infty \in (-1/4, 1/4)}$. Define a sequence $(u_0^\nu, u_1^\nu, u_2^\nu)$ of $(\widehat{J}_0, \widehat{J}_1, \widehat{J}_2, i)$-holomorphic size-$(\frac{1}{4}, \theta^\nu)$

squiggly strip quilts, with

$$\theta^\nu\colon [-\tfrac{1}{4},\tfrac{1}{4}] \to (0,\tfrac{1}{2}], \qquad \theta^\nu(s) := \tfrac{1}{2\pi}\arcsin(\tfrac{1}{2}e^{2\pi(s+s^\nu)}),$$

by:

$$u^\nu_\ell(s,t) := u_\ell(s+s^\nu,t).$$

The energy $\int_{B_{1/8}(0)} \frac{1}{2}|\mathrm{d}\underline{u}^\nu|^2$ is bounded by the energy of $\underline{v}$, and by assumption, the gradient $|\mathrm{d}\underline{u}^\nu(0,t^\nu)|$ tends to ∞. In the following sublemma we establish the last hypothesis needed to apply Theorem 3.3.1.

Sublemma 2.1.7. *The functions $\theta^\nu(s) = \frac{1}{2\pi}\arcsin(\frac{1}{2}e^{2\pi(s+s^\nu)})$ obediently shrink to zero as $\nu\to\infty$.*

Proof of Sublemma 2.1.7. The convergence $s^\nu \to -\infty$ implies $\frac{1}{2}e^{2\pi(s+s^\nu)} \to 0$ in $\mathcal{C}^0$, so the equality $\arcsin(0) = 0$ implies the $\mathcal{C}^0$-convergence of θ^ν to zero.

To check the second condition for obedient shrinking, fix $k \geq 1$ and note that $\frac{\mathrm{d}^k\theta^\nu}{\mathrm{d}s^k}(s) = \frac{\mathrm{d}^k\theta}{\mathrm{d}s^k}(s+s^\nu)$, with $\theta(s) = \frac{1}{2\pi}\arcsin(\frac{1}{2}e^{2\pi s})$ as above. The derivative $\frac{\mathrm{d}^k\theta}{\mathrm{d}s^k}(s)$ is a linear combination of the functions $f_\ell(s) := (4-e^{4\pi s})^{-(\ell-1/2)}e^{4\pi(\ell-1/2)s}$ for $\ell = 1,\ldots,m$. (This can be seen by induction starting from $\frac{\mathrm{d}\theta(s)}{\mathrm{d}s} = (4-e^{4\pi s})^{-1/2}e^{2\pi s}$.) This decomposition, the inequality $\arcsin(x) \geq x$ for $x\in[0,1]$, and the convergence $s^k \to -\infty$ allows us to establish the second condition:

$$\begin{aligned}
\sup_{\nu\in\mathbb{N}} \frac{\max_{s\in[-1/4,1/4]}|f_\ell(s)|}{\min_{s\in[-1/4,1/4]}\theta^\nu(s)} &\leq \sup_{\nu\in\mathbb{N}} \frac{\exp(4\pi(\ell-\frac{1}{2})(s^\nu+1/4))}{\frac{1}{4\pi}\exp(2\pi(s^\nu-1/4))}\\
&= \sup_{\nu\in\mathbb{N}} 4\pi\exp(4\pi((\ell-1)s^\nu+\tfrac{1}{4}))\\
&\leq 4\pi\exp(\pi).
\end{aligned}$$

The arcsine function extends to a holomorphic function $\arcsin\colon \overline{B}_1(0)\to\mathbb{C}$ by the power series $\arcsin(z) := \sum_{k=0}^\infty \frac{\binom{2k}{k}z^{2k+1}}{4^k(2k+1)}$, so f^ν extends to a holomorphic function F^ν from $[-1/4,1/4]^2$ to $\mathbb{C}$. Since the functions $\frac{1}{2}e^{2\pi(z+s^\nu)}$ tend $\mathcal{C}^\infty$ to zero and since $\arcsin(0) = 0$, the extensions F^ν also tend $\mathcal{C}^\infty$ to zero. □

Part (2) of Theorem 3.3.1 now implies the inequality $\liminf_{\nu\to\infty}\int_{B_{1/8}(0)}\frac{1}{2}|\mathrm{d}\underline{u}^\nu|^2 > 0$, which contradicts (2.13). □

Proof of Lemma 2.1.3. First, note that the domain $[1/2-\theta(s),1/2+\theta(s)]\cup[1-\theta(s),1+\theta(s)]$ of γ^1_s has total length $4\theta(s) = \frac{2}{\pi}\arcsin\left(\frac{1}{2}e^{2\pi s}\right)$, which converges to 0 as $s\to-\infty$. Hence the gradient bounds of Lemma 2.1.4 immediately imply that the L^2-length of γ^1_s converges to zero as $s\to-\infty$. Moreover, these gradient bounds imply that to show the L^2-lengths of γ^0_s,γ^2_s converge to zero, it suffices to fix an arbitrary $\epsilon>0$ and show that the L^2-lengths of $\gamma^0_s|_{[\epsilon,1/2-\epsilon]},\gamma^2_s|_{[1/2+\epsilon,1-\epsilon]}$ converge to zero as $s\to-\infty$.

Fix $\epsilon>0$. We will show that the L^2-length of $\gamma^0_s|_{[1/2+\epsilon,1-\epsilon]}$ converges to zero as $s\to-\infty$; the proof for γ_2 is similar. Choose s_0 so that the domain of γ^0_s contains $[1/2+\epsilon/2,1-\epsilon/2]$ for all $s\leq s_0$. Now the $\mathcal{C}^0$-bound on $|\mathrm{d}v_0|$ from Lemma 2.1.4 induces a $\mathcal{C}^m$-bound on

$v_0|_{(-\infty,s_0-1]\times[1/2+\epsilon,1-\epsilon]}$ for any $m \geq 0$. Indeed, we can apply the interior elliptic estimates (e.g. [AbbHo, §6.3]) on each of the precompactly-nested domains

$$[s_0 - k - 1, s_0 - k] \times [1/2 + \epsilon, 1 - \epsilon] \subset [s_0 - k - 2, s_0 - k + 1] \times [1/2 + \epsilon/2, 1 - \epsilon/2]$$

for $k \in \mathbb{N}$. Since for different k these domains are translations of one other, the constants in the elliptic estimates are independent of k, and thus yield the desired C^m-bounds.

For $s \leq s_0$, define

$$\Phi(s) := \tfrac{1}{2}\int_{-\infty}^{s}\int_{1/2+\epsilon}^{1-\epsilon} \left(|\partial_s v_0|^2 + |\partial_t v_0|^2\right).$$

Then $\Phi\colon (-\infty, s_0] \to [0,\infty)$ is nondecreasing with $\lim_{s\to-\infty}\Phi(s) = 0$ and

$$\Phi'(s) = \tfrac{1}{2}\int_{1/2+\epsilon}^{1-\epsilon} (|\partial_s v_0(s,\tau)|^2 + |\partial_t v_0(s,\tau)|^2)\,\mathrm{d}\tau = \int_{1/2+\epsilon}^{1-\epsilon} |\partial_t v_0(s,\tau)|^2\,\mathrm{d}\tau,$$
$$\Phi''(s) = 2\int_{1/2+\epsilon}^{1-\epsilon} \langle \partial_s v_0(s,\tau), \nabla^2_{\mathrm{LC},s} v_0(s,\tau)\rangle\,\mathrm{d}\tau,$$

where in the last quantity we are using the Levi-Civita connection with respect to the metric g_0 defined in (2.3). By the previous paragraph, there exists a constant $c > 0$ so that $\Phi''(s) \leq c$ for all $s \leq s_0 - 1$. Now for any fixed $\delta > 0$ we can choose $s_1 \leq s_0 - 1$ such that $\Phi(s_1) \leq \delta^2/4c$. For $s \leq s_1$, we obtain:

$$\frac{\delta^2}{4c} \;\geq\; \Phi(s_1) \;\geq\; \Phi(s) - \Phi(s - \tfrac{\delta}{2c}) \;=\; \int_{s-\delta/2c}^{s} \Phi'(\sigma)\,\mathrm{d}\sigma \;\geq\; \frac{\delta}{2c}(\Phi'(s) - \tfrac{\delta}{2}),$$

where the last step uses the bound on Φ'' to deduce $\Phi'(\sigma) \geq \Phi'(s) - c|s-\sigma|$. This inequality can be rearranged to yield $\Phi'(s) \leq \delta$ for all $s \leq s_1$, and thus proves $\lim_{s\to-\infty}\Phi'(s) = 0$. Since $\Phi'(s)$ is equal to $\|\frac{\mathrm{d}}{\mathrm{d}t}\gamma_s^0\|^2_{L^2([1/2+\epsilon,1-\epsilon])}$ and since $|\frac{\mathrm{d}}{\mathrm{d}t}\gamma_s^0|$ is uniformly bounded, we have now shown that the L^2-norm of γ_s^0 converges to zero as $s \to -\infty$.

The Cauchy–Schwarz inequality implies that the L^1-norm of $\frac{\mathrm{d}}{\mathrm{d}t}\gamma_0^s$ — i.e. the length $\ell(\gamma_1^s)$ — also tends to zero as $s \to -\infty$. □

2.1.2 An isoperimetric inequality and the proof of removal of singularity

In this subsection, we prove Theorem 2.1.2. The crucial inputs will be Lemma 2.1.3 from §2.1.1 together with the following isoperimetric inequality for the energy on $(-\infty, s_0] \times \mathbb{R}/\mathbb{Z}$,

$$E(\underline{v}; s_0) := \int_{(-\infty,s_0]\times\mathbb{R}/\mathbb{Z}} \tfrac{1}{2}|\mathrm{d}\underline{v}|^2\,\mathrm{d}s\mathrm{d}t.$$

Lemma 2.1.8. *There exists $C > 0$ such that the following inequality holds for all $s \leq 0$:*

$$E(\underline{v}; s) \leq C \sum_{i\in\{0,1,2\}} \ell(\gamma_s^i)^2.$$

We defer the proof to later in §2.1.2; now, we turn to the proof of removal of singularity. Throughout this subsection we denote

$$M_{0112} := M_0^- \times M_1 \times M_1^- \times M_2, \qquad M_{02} := M_0^- \times M_2.$$

Proof of Theorem 2.1.2.

Step 1. *There exist $C_1, C_2 > 0$ such that the inequality $E(\underline{v}; s) \leq C_1 \exp(C_2 s)$ holds for all $s \leq 0$.*

Fix $s \leq 0$. The following inequality follows from Lemma 2.1.8:

$$E(\underline{v}; s) \overset{\text{Lem. 2.1.8}}{\leq} C \sum_{\ell \in \{0,1,2\}} \ell(v_\ell(s,-))^2 \leq \frac{C}{2} \left(\textstyle\int_0^1 |\mathrm{d}v(s,t)| \,\mathrm{d}t \right)^2 \leq \frac{C}{2} \textstyle\int_0^1 |\mathrm{d}v(s,t)|^2 \,\mathrm{d}t$$
$$= C \frac{\mathrm{d}}{\mathrm{d}s}(E(\underline{v}; s)).$$

Manipulating this inequality and integrating from s to 0, we obtain $E(\underline{v}; s) \leq E(\underline{v}; 0) \exp(s/C)$.

Step 2. *The limit $\lim_{s \to -\infty} v_0(s, -)$ exists in $\mathcal{C}^0([5/8, 7/8], M_0)$.*

Fix a $\mathcal{C}^1$ embedding $i \colon M_0 \to \mathbb{R}^N$; we will show that $\Lambda := \lim_{s \to -\infty}(i \circ v_0|_{[5/8,7/8]})$ exists in $\mathcal{C}^0$.

We begin by showing that Λ exists in L^2, where $L^2([5/8, 7/8], \mathbb{R}^N)$ is defined using the Euclidean metric on $\mathbb{R}^N$. Fix $s_2 \leq s_1 \leq 0$. Cauchy–Schwarz implies the following inequality:

$$\begin{aligned} \|(i \circ v_0)(s_1, -) - (i \circ v_0)(s_2, -)\|_{L^2([5/8,7/8])} &= \left(\int_{5/8}^{7/8} \left| \int_{s_2}^{s_1} \partial_s (i \circ v_0) \,\mathrm{d}s \right|^2 \mathrm{d}t \right)^{1/2} \\ &\leq (s_1 - s_2)^{1/2} \left(\int_{5/8}^{7/8} \int_{s_2}^{s_1} |\partial_s (i \circ v_0)|^2_{g_{\mathrm{euc}}} \,\mathrm{d}s\mathrm{d}t \right)^{1/2}. \end{aligned} \tag{2.14}$$

Since M_0 is compact, there exists a constant of equivalence $\mu > 0$ for the norms induced by g_{M_0} and $i^* g_{\mathrm{euc}}$, so (2.14) yields the following:

$$\begin{aligned} \|(i \circ v_0)(s_1, -) - (i \circ v_0)(s_2, -)\|_{L^2([5/8,7/8])} &\overset{(2.14)}{\leq} \mu (s_1 - s_2)^{1/2} \left(\int_{5/8}^{7/8} \int_{s_2}^{s_1} |\partial_s v_0|^2_{g_{M_0}} \,\mathrm{d}s\mathrm{d}t \right)^{1/2} \\ &\overset{\text{Step 1}}{\leq} \mu\, C_1^{1/2} (s_1 - s_2)^{1/2} \exp(C_2 s_1 / 2) \\ &=: C_3 (s_1 - s_2)^{1/2} \exp(C_2 s_1/2). \end{aligned} \tag{2.15}$$

Write $s_2 = (m + \epsilon) s_1$ for $m \in \mathbb{N}$ and $\epsilon \in [0, 1)$. We have:

$$\begin{aligned} &\|(i \circ v_0)(s_1, -) - (i \circ v_0)((m + \epsilon)s_1, -)\|_{L^2([5/8,7/8])} \\ &\qquad \leq \|(i \circ v_0)(m s_1, -) - (i \circ v_0)((m + \epsilon)s_1, -)\|_{L^2([5/8,7/8])} \\ &\qquad\qquad + \sum_{j=1}^{m-1} \|(i \circ v_0)(j s_1, -) - (i \circ v_0)((j + 1)s_1, -)\|_{L^2([5/8,7/8])} \end{aligned} \tag{2.16}$$

$$\begin{aligned} &\overset{(2.15)}{\leq} C_3 |s_1|^{1/2} \sum_{j=1}^{m} \exp(j C_2 s_1/2) \\ &\leq \frac{C_3 |s_1|^{1/2} \exp(C_2 s_1/2)}{1 - \exp(C_2 s_1/2)}. \end{aligned}$$

Define $f(s) := |\frac{\mathrm{d}}{\mathrm{d}t}(i \circ v_0)(s,-)|_{L^2([5/8,7/8])}$. This quantity tends to zero as $s \to -\infty$:

$$\limsup_{s\to-\infty} f(s) \leq \limsup_{s\to-\infty} \mu|\tfrac{\mathrm{d}}{\mathrm{d}t}v_0(s,-)|_{L^2([5/8,7/8])} \overset{\text{Lem. 2.1.3}}{=} 0.$$

We can now show that Λ exists in $W^{1,2}$: We have

$$\begin{aligned}
|(i\circ v_0)(s_1,-)-(i\circ v_0)(s_2,-)|_{W^{1,2}([5/8,7/8])} &\leq |(i\circ v_0)(s_1,-)-(i\circ v_0)(s_2,-)|_{L^2([5/8,7/8])} \\
&\quad + f(s_1)+f(s_2) \\
&\overset{(2.16)}{\leq} \frac{C_3|s_1|^{1/2}\exp(C_2 s_1/2)}{1-\exp(C_2 s_1/2)} + f(s_1)+f(s_2),
\end{aligned}$$

which implies the equality

$$\limsup_{s_1\to-\infty} \sup_{s_2\in(-\infty,s_1]} |(i\circ v_0)(s_1,-)-(i\circ v_0)(s_2,-)|_{W^{1,2}([5/8,7/8])} = 0.$$

Since $W^{1,2}([5/8,7/8],\mathbb{R}^N)$ is complete, Λ exists in $W^{1,2}$. The Sobolev embedding $W^{1,2}\hookrightarrow C^0$ for one-dimensional domains now implies that Λ exists in C^0.

Step 3. *We prove Theorem 2.1.2.*

By Lemma 2.1.3, the first claim of Theorem 2.1.2 would follow from the existence of the limits

$$\begin{aligned}
\Lambda_0 &:= \lim_{s\to-\infty} v_0(s,\tfrac{3}{4}), & \Lambda_2 &:= \lim_{s\to-\infty} v_2(s,\tfrac{1}{4}), \\
\Lambda_1 &:= \lim_{s\to-\infty} v_1(s,\tfrac{1}{2}), & \Lambda_1' &:= \lim_{s\to-\infty} v_1(s,1).
\end{aligned}$$

It follows from Step 2 that Λ_0 exists, and an analogous argument shows that Λ_2 exists. It remains to show that Λ_1, Λ_1' exist.

To show that Λ_1 exists, we will show convergence of the path

$$\gamma\colon s \mapsto (v_0(s,\tfrac{1}{2}+\theta(s)), v_1(s,\tfrac{1}{2}), v_1(s,\tfrac{1}{2}), v_2(s,\tfrac{1}{2}-\theta(s))$$

as $s\to-\infty$. This path takes values in $M_0\times\Delta_{M_1}\times M_2$ and $\lim_{s\to-\infty} d_{M_{0112}}(\gamma(s), L_{01}\times L_{12}) = 0$ (by Lemma 2.1.4), so the distances $d_{M_{0112}}(\gamma(s), L_{01}\times_{M_1} L_{12})$ converge to zero. Hence there exists a path $\beta\colon(-\infty,0]\to L_{01}\times_{M_1}L_{12}$ satisfying the equality

$$\lim_{s\to-\infty} d_{M_{0112}}(\gamma(s),\beta(s)) = 0. \tag{2.17}$$

(Indeed, define β by choosing a tubular neighborhood U of $L_{01}\times_{M_1}L_{12}$, and compose γ with the projection $U\to L_{01}\times_{M_1}L_{12}$.) We will show that $\lim_{s\to-\infty}\gamma(s)$ exists by showing that $\lim_{s\to-\infty}\beta(s)$ exists.

Lemma 2.1.3, the existence of Λ_0 and Λ_2, and (2.17) imply that $x_{02} := \lim_{s\to-\infty}\pi_{02}(\beta(s))$ exists. Since π_{02} restricts to an immersion of $L_{01}\times_{M_1}L_{12}$ into M_{02}, there exist finitely many preimages $x_{0112}^1,\ldots,x_{0112}^k$ of x_{02} in $L_{01}\times_{M_1}L_{12}$. Choose $\epsilon>0$ small enough that the preimage of $B_\epsilon(x_{02})$ under $\pi_{02}|_{L_{01}\times_{M_1}L_{12}}$ consists of k connected components $U^1,\ldots,U^k$, with x_{0112}^j contained in U^j. Now choose $s_0\in(-\infty,0]$ such that $\pi_{02}(\beta((-\infty,s_0]))$ is contained in $B_\epsilon(x_{02})$. The image $\beta((-\infty,s_2])$ must then be contained in a single U_j. If $(s_\nu),(s_\nu')$ are

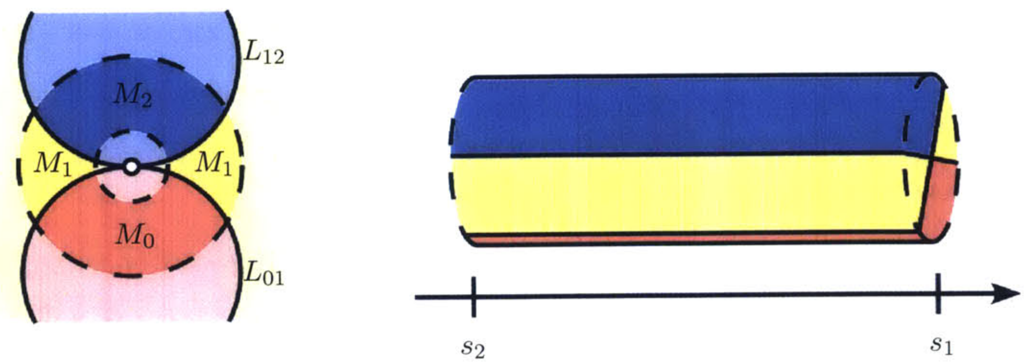


Figure 2-3: The start of our argument for Lemma 2.1.8 is to restrict an inverted figure eight to an annulus centered at the singular point (the portion in the left figure between the dotted circles), then reparametrize to a quilted tube with straight seams (the tubular part of the boundary of the cylinder on the right). Next, we piecewise-smoothly extend to the interior of the cylinder.

sequences with limit $-\infty$ such that $x_{0112}^{j_1} := \lim_{\nu\to\infty}\beta(s_\nu)$ and $x_{0112}^{j_2} := \lim_{\nu\to\infty}\beta(s'_\nu)$ exist, then j_1 and j_2 must be equal; since $L_{01}\times_{M_1} L_{12}$ is compact, this is enough to conclude that $\lim_{s\to-\infty}\beta(s)$ exists. As noted above, this is enough to conclude the first statement of Theorem 2.1.2.

The points $(\Lambda_0, \Lambda_1, \Lambda_1, \Lambda_2)$ and $(\Lambda_0, \Lambda'_1, \Lambda'_1, \Lambda_2)$ are lifts in $L_{01}\times_{M_1} L_{12}$ of (Λ_0, Λ_2), so if the projection from $L_{01}\times_{M_1} L_{12}$ to M_{02} is injective, then Λ_1, Λ'_1 are the same point. $\square$

Our proof of Lemma 2.1.8 is an adaptation to the quilted setting of [McSa, Lemma 4.5.1], which is an isoperimetric inequality for the energy near an interior point of a J-holomorphic curve. Their argument went like this: restricting the map to an annulus, then reparametrizing, yields a map defined on the curved part of the boundary of a cylinder. By a lengths-go-to-zero result analogous to our Lemma 2.1.3, they extend this map to the entire cylinder. Their result now follows from Stokes' theorem, along with the isoperimetric inequality for the symplectic area applied to the top and bottom caps of the cylinder. The difficulty in adapting this result to the quilted setting is in the extension to the cylinder (see Figure 2-3 for an illustration of the setup); the key will be the consequences of cleanly-immersed composition recorded in the following lemma.

Lemma 2.1.9. *There exist $C > 0, \epsilon > 0$ such that:*

(i) If $x_{02}, y_{02} \in L_{01}\circ L_{12}$ have lifts

$$x, x' \in \pi_{02}^{-1}\{x_{02}\}\cap(L_{01}\times_{M_1} L_{12}), \qquad y, y' \in \pi_{02}^{-1}\{y_{02}\}\cap(L_{01}\times_{M_1} L_{12})$$

with small distances

$$\max\{d_{M_{0112}}(x,y), d_{M_{0112}}(x',y')\} \le \epsilon,$$

then there exists a smooth path $\gamma_{02}\colon [0,1]\to M_{02}$ with image in $L_{01}\circ L_{12}$ and smooth lifts $\gamma, \gamma'\colon [0,1]\to L_{01}\times_{M_1} L_{12}$ that have bounded lengths

$$\ell(\gamma_{02}) + \ell(\gamma) + \ell(\gamma') \le C\, d_{M_{02}}(x_{02}, y_{02})$$

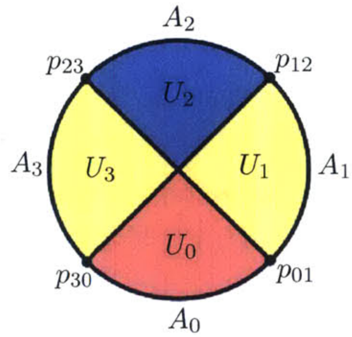


Figure 2-4: The domains used in the proof of Lemma 2.1.8.

and satisfy $\gamma(0) = x$, $\gamma(1) = y$, $\gamma'(0) = x'$, *and* $\gamma'(1) = y'$.

(ii) For $x, x' \in L_{01} \times_{M_1} L_{12}$ *with* $d_{M_{02}}(\pi_{02}(x), \pi_{02}(x')) \le \epsilon$, *there exists a point* $y_{02} \in L_{01} \circ L_{12}$ *and preimages* $y, y' \in \pi_{02}^{-1}(y_{02}) \cap L_{01} \times_{M_1} L_{12}$ *such that the following inequality holds:*

$$d_{M_{02}}(\pi_{02}(x'), y_{02}) + d_{M_{02}}(\pi_{02}(x), y_{02}) + d_{M_{0112}}(x, y) + d_{M_{0112}}(x', y') \\ \le C\, d_{M_{02}}(\pi_{02}(x), \pi_{02}(x')).$$

We will give only a brief sketch, since a formal proof is no more enlightening. The key is that the cleanly-immersed hypothesis implies that any two branches of $L_{01} \circ L_{12}$ meet like two vector subspaces.

(i) If x, x', y, y' lie in the same local branch of $L_{01} \circ L_{12}$, then the conclusion is immediate. Otherwise, x and y lie in one branch, and x' and y' lie in another. Represent these branches as open subsets of vector subspaces $V, V' \subset \mathbb{R}^N$. Then x_{02}, y_{02} lie in $V \cap V'$, and we may define γ_{02} to be a path in $V \cap V'$ from x_{02} to y_{02} and γ (resp. γ') to be the lift to the portion of $L_{01} \times_{M_1} L_{12}$ corresponding to V (resp. to V').

(ii) If x, x' lie in the same local branch of $L_{01} \circ L_{12}$, the conclusion is again immediate. Otherwise, represent the branches containing x, x' as open subsets of $V, V' \subset \mathbb{R}^N$. Set y_{02} to be the nearest point in $V \cap V'$ to x, and let y (resp. y') be the lift to the portion of $L_{01} \times_{M_1} L_{12}$ corresponding to V (resp. to V').

Proof of Lemma 2.1.8.

Step 1. *We prove Lemma 2.1.8 up to an extension result, which we defer to Steps 2 and 3.* It suffices to prove the lemma for $s \le s_0 \le 0$, where s_0 is chosen so that $\sup_{s \le s_0} \ell(\gamma_s^i)$, $i \in \{0, 1, 2\}$ is bounded by a constant $\delta > 0$ to be determined later. As illustrated in Figure 2-4, partition the unit circle $S_1(0)$ into four segments by

$$A_0 := \{(x, y) \in S_1(0) \mid y \le x,\ y \le -x\}, \qquad A_1 := \{(x, y) \in S_1(0) \mid x \ge y,\ x \ge -y\},$$
$$A_2 := \{(x, y) \in S_1(0) \mid y \ge x,\ y \ge -x\}, \qquad A_3 := \{(x, y) \in S_1(0) \mid x \le y,\ x \le -y\}$$

and set $p_{i(i+1)} := A_i \cap A_{i+1}$ for $i \in \mathbb{Z}/4\mathbb{Z}$. Given s_1, s_2 with $s_2 < s_1 \le s_0$, define maps

$\sigma_i \colon A_i \times [s_2, s_1] \to M_i$, $i \in \{0,1,2,3\}$ (where we set $M_3 := M_1$) like so:

$$\begin{aligned}
\sigma_0(\exp(2\pi i t), s) &:= v_0\big(s, \tfrac{1}{2} + \theta(s) + 4(\tfrac{1}{2} - 2\theta(s))(t - \tfrac{5}{8})\big),\\
\sigma_1(\exp(2\pi i t), s) &:= v_1\big(s, 8\theta(s)t\big),\\
\sigma_2(\exp(2\pi i t), s) &:= v_2\big(s, \theta(s) + 4(\tfrac{1}{2} - 2\theta(s))(t - \tfrac{1}{8})\big),\\
\sigma_3(\exp(2\pi i t), s) &:= v_1\big(s, \tfrac{1}{2} + 8\theta(s)(t - \tfrac{1}{2})\big),
\end{aligned}$$

where we take $t \in [-1/8, 7/8]$. These maps satisfy the seam condition

$$(\sigma_i(p_{i(i+1)}), \sigma_{i+1}(p_{i(i+1)})) \in L_{i(i+1)}, \qquad i \in \mathbb{Z}/4\mathbb{Z},$$

where we set $L_{23} := L_{12}^T$, $L_{30} := L_{01}^T$.

In order to apply Stokes' theorem, we will extend the maps σ_i to the following four quadrants of the closed unit disk:

$$\begin{aligned}
U_0 &:= \{(x,y) \in \overline{B}(0,1) \mid y \le x,\ y \le -x\}, &\quad U_1 &:= \{(x,y) \in \overline{B}(0,1) \mid x \ge y,\ x \ge -y\},\\
U_2 &:= \{(x,y) \in \overline{B}(0,1) \mid y \ge x,\ y \ge -x\}, &\quad U_3 &:= \{(x,y) \in \overline{B}(0,1) \mid x \le y,\ x \le -y\}.
\end{aligned}$$

Choose $s_2 = t_0 < t_1 < \cdots < t_k = s_1$ such that for every j, the diameters of the images $\sigma_i(A_i \times [t_j, t_{j+1}])$ are bounded by δ. As long as δ is small enough, Steps 2 and 3 below allow us to extend σ_i to a continuous map $\widetilde{\sigma}_i \colon U_i \times [s_2, s_1] \to M_i$ that is smooth on $U_i \times [t_j, t_{j+1}]$, such that the extended maps satisfy the Lagrangian seam conditions

$$(\widetilde{\sigma}_i(p,s), \widetilde{\sigma}_{i+1}(p,s)) \in L_{i(i+1)} \qquad \forall\, p \in U_i \cap U_{i+1},\ s \in [s_2, s_1] \tag{2.18}$$

Indeed, use Step 2 to define the maps $\widetilde{\sigma}_i$ on the slices $U_i \times \{t_j\}$, then use Step 3 to extend $\widetilde{\sigma}_i$ to all of $U_i \times [s_2, s_1]$.

Since $\omega_0, \omega_1, \omega_2$ are closed, Stokes' theorem yields the following:

$$E(\underline{v}; [s_2, s_1] \times \mathbb{R}/\mathbb{Z}) \le \sum_{i \in \{1,2\}} \Big| \sum_{j \in \{0,1,2,3\}} \textstyle\int_{U_j \times \{s_i\}} \sigma_j^* \omega_j \Big| \le C \displaystyle\sum_{i \in \{1,2\}} \sum_{j \in \{1,2,3\}} \ell(\gamma_{s_i}^j)^2,$$

where in the first inequality we have used the seam conditions (2.18), and in the second inequality we have used the isoperimetric inequality for the symplectic area [McSa, Theorem 4.4.1]. Taking the limit as s_2 goes to $-\infty$ and applying Lemma 2.1.3 yields the conclusion of the lemma.

Throughout the final two steps, the constants C_i will depend only on the geometry of L_{01}, L_{12}.

Step 2. *There exist $C > 0$, $\kappa_0 > 0$ so that if $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ are smooth maps with*

$$\sigma_i \colon A_i \to M_i, \qquad (\sigma_i(p_{i(i+1)}), \sigma_{i+1}(p_{i(i+1)})) \in L_{i(i+1)}, \qquad \kappa := \max_{i \in \{0,1,2,3\}} \operatorname{diam} \sigma_i(A_i) \le \kappa_0,$$

then there exist extensions $\widetilde{\sigma}_i \colon U_i \to M_i$ of σ_i such that:

$$\begin{gathered}
(\widetilde{\sigma}_i(p), \widetilde{\sigma}_{i+1}(p)) \in L_{i(i+1)} \quad \forall\, p \in U_i \cap U_{i+1},\\
\max_{i \in \{0,1,2,3\}} \ell(\widetilde{\sigma}_i|_{\partial U_i}) + \max_{i \in \{0,1,2,3\}} \operatorname{diam} \widetilde{\sigma}_i(U_i) \le C\kappa.
\end{gathered}$$

The points

$$z := (\sigma_0(p_{01}), \sigma_1(p_{01}), \sigma_1(p_{12}), \sigma_2(p_{12})), \qquad z' := (\sigma_0(p_{30}), \sigma_3(p_{30}), \sigma_3(p_{23}), \sigma_2(p_{23}))$$

lie in $L_{01} \times L_{12}$. Since the intersection $(L_{01} \times L_{12}) \cap (M_0 \times \Delta_{M_1} \times M_2)$ defining $L_{01} \times_{M_1} L_{12}$ is transverse, there are points $x, x' \in L_{01} \times_{M_1} L_{12}$ that are close to z resp. z',

$$d_{M_{0112}}(x, z) \leq C_1\kappa, \qquad d_{M_{0112}}(x', z') \leq C_1\kappa, \tag{2.19}$$

for a uniform constant $C_1 > 0$. The triangle inequality bounds the distance between the projections of z, z':

$$\begin{aligned} d_{M_{02}}(\pi_{02}(x), \pi_{02}(x')) &\leq d_{M_{02}}(\pi_{02}(x), \pi_{02}(z)) + d_{M_{02}}(\pi_{02}(z), \pi_{02}(z')) + d_{M_{02}}(\pi_{02}(z'), \pi_{02}(x')) \\ &\leq 2(C_1 + 1)\kappa. \end{aligned}$$

As long as κ_0 is chosen to be small enough, it follows from Lemma 2.1.9(ii) that there exist lifts $y, y' \in L_{01} \times_{M_1} L_{12}$ of a single point $y_{02} \in L_{01} \circ L_{12}$ with small distances to z resp. z':

$$d_{M_{0112}}(x, y) \leq C_2\kappa, \qquad d_{M_{0112}}(x', y') \leq C_2\kappa, \tag{2.20}$$

where $C_2 > 0$ is another constant. We can now define the extensions $\widetilde{\sigma}_i$ at the origin:

$$(\widetilde{\sigma}_0(0), \widetilde{\sigma}_1(0), \widetilde{\sigma}_1(0), \widetilde{\sigma}_2(0)) := y, \qquad (\widetilde{\sigma}_0(0), \widetilde{\sigma}_3(0), \widetilde{\sigma}_3(0), \widetilde{\sigma}_2(0)) := y'.$$

Inequalities (2.19) and (2.20) and the triangle inequality yield:

$$d_{M_{0112}}(y, z) \leq (C_1 + C_2)\kappa, \qquad d_{M_{0112}}(y', z') \leq (C_1 + C_2)\kappa.$$

The local triviality of smooth submanifolds implies that there exists a constant $C_3 > 0$ such that after redefining κ_0 if necessary, we may extend the maps $\widetilde{\sigma}_i$ to the set $\{(a, b) \in \overline{B}(0, 1) \mid b = \pm a\}$ such that the seam conditions (2.18) hold and the length of the loop $\widetilde{\sigma}_i|_{\partial U_i}$ is bounded by $C_3\kappa$. Once more redefining κ_0 if necessary, we may extend each map $\widetilde{\sigma}_i$ to U_i in such a way that the diameter of $\widetilde{\sigma}_i(U_i)$ is bounded by $C_4\kappa$ for $C_4 > 0$ another constant.

Step 3. *There exists $\lambda > 0$ such that the following holds. Assume that $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ are smooth maps and $a < b$ are real numbers with:*

$$\begin{gathered} \sigma_i \colon A_i \times [a, b] \cup U_i \times \{a, b\} \to M_i, \qquad \max_{i \in \{0,1,2,3\}} \operatorname{diam} \operatorname{im} \sigma_i \leq \lambda, \\ (\sigma_i(q), \sigma_{i+1}(q)) \in L_{i(i+1)} \qquad \forall\, q \in \big(p_{i(i+1)} \times [a, b]\big) \cup \big((U_i \cap U_{i+1}) \times \{a, b\}\big). \end{gathered}$$

Then each σ_i can be extended to a smooth map $\widetilde{\sigma}_i \colon U_i \times [a, b] \to M_i$ such that the following seam conditions hold:

$$(\widetilde{\sigma}_i(q), \widetilde{\sigma}_{i+1}(q)) \in L_{i(i+1)} \qquad \forall\, q \in (U_0 \cap U_1) \times [a, b].$$

Define $x, x', y, y' \in L_{01} \times_{M_1} L_{12}$ like so:

$$\begin{aligned} x &:= (\sigma_0, \sigma_1, \sigma_1, \sigma_2)(0, a), \qquad & x' &:= (\sigma_0, \sigma_3, \sigma_3, \sigma_2)(0, a), \\ y &:= (\sigma_0, \sigma_1, \sigma_1, \sigma_2)(0, b), \qquad & y' &:= (\sigma_0, \sigma_3, \sigma_3, \sigma_2)(0, b). \end{aligned}$$

Then $\pi_{02}(x) = \pi_{02}(x')$ and $\pi_{02}(y) = \pi_{02}(y')$, and x resp. x' are close to y resp. y':

$$d_{M_{0112}}(x, y) \leq 4\lambda, \qquad d_{M_{0112}}(x', y') \leq 4\lambda.$$

It follows from Lemma 2.1.9(i) that as long as λ is chosen to be small enough, there exists a path $\gamma_{02}\colon [a, b] \to L_{01} \circ L_{12}$ and lifts $\gamma, \gamma'\colon [a, b] \to L_{01} \times_{M_1} L_{12}$ from x to y resp. from x' to y' of small lengths:

$$\ell(\gamma) + \ell(\gamma') \leq C_5\lambda$$

for $C_5 > 0$ a constant. Define $\widetilde{\sigma}_0, \widetilde{\sigma}_1, \widetilde{\sigma}_2, \widetilde{\sigma}_3$ on $\{0\} \times [a, b]$ like so:

$$(\widetilde{\sigma}_0, \widetilde{\sigma}_1, \widetilde{\sigma}_1, \widetilde{\sigma}_2)(0, t) := \gamma(t), \qquad (\widetilde{\sigma}_0, \widetilde{\sigma}_3, \widetilde{\sigma}_3, \widetilde{\sigma}_2)(0, t) := \gamma'(t).$$

The diameter of the loop $(\widetilde{\sigma}_0, \widetilde{\sigma}_1)|_{\partial((U_0 \cap U_1) \times [a,b])}$ is bounded by $2(C_5 + 1)\lambda$, so by redefining λ if necessary, we may extend $(\widetilde{\sigma}_0, \widetilde{\sigma}_1)$ to a map $(U_0 \cap U_1) \times [a, b] \to M_0^- \times M_1$ with small diameter:

$$\operatorname{diam}\left((\widetilde{\sigma}_0, \widetilde{\sigma}_1)((U_0 \cap U_1) \times [a, b])\right) \leq C_6\lambda$$

for $C_6 > 0$ a constant. Extend $(\widetilde{\sigma}_1, \widetilde{\sigma}_2)$, $(\widetilde{\sigma}_2, \widetilde{\sigma}_3)$, $(\widetilde{\sigma}_3, \widetilde{\sigma}_0)$ to $(U_1 \cap U_2) \times [a, b]$, $(U_2 \cap U_3) \times [a, b]$, $(U_3 \cap U_0) \times [a, b]$ in the same fashion. Finally, $\widetilde{\sigma}_i|_{\partial(U_i \times [a,b])}$ is a map to M_i from a domain homeomorphic to S^2, and its diameter is small:

$$\operatorname{diam}\left(\widetilde{\sigma}_i(\partial(U_i \times [a, b]))\right) \leq (2C_6 + 1)\lambda.$$

Redefining λ if necessary, we may extend $\widetilde{\sigma}_i$ to all of $U_i \times [a, b]$. $\square$

2.2 Convergence modulo bubbling for strip-shrinking

The proof of Gromov Compactness Theorem 3.3.1 relies on $\mathcal{C}^k$-compactness in the presence of a uniform gradient bound. This result is based on a strengthening of the strip-shrinking analysis of [WeWo4] from $H^2 \cap W^{1,4}$-convergence to $\mathcal{C}^k$-convergence; we also allow the domain to be equipped with nonstandard complex structures and the geometric composition $L_{01} \circ L_{12}$ to be immersed, rather than embedded. The purpose of this section is to establish convergence mod bubbling in Theorem 2.2.1, deferring the δ-independent Sobolev estimate Lemma 2.2.8 to §2.2.2.

Fix for §2.2 closed symplectic manifolds M_0, M_1, M_2 and compact Lagrangians $L_{01} \subset M_0^- \times M_1$, $L_{12} \subset M_1^- \times M_2$ with immersed composition as defined in the beginning of §2.1.

For convenience, we will denote by (M_{02}, ω_{02}), $(M_{0211}, \omega_{0211})$ the symplectic manifolds

$$(M_{0211}, \omega_{0211}) := M_0 \times M_2^- \times M_1^- \times M_1 = (M_0 \times M_2 \times M_1 \times M_1, \omega_0 \oplus (-\omega_2) \oplus (-\omega_1) \oplus \omega_1),$$
$$(M_{02}, \omega_{02}) := M_0^- \times M_2 = (M_0 \times M_2, (-\omega_0) \oplus \omega_2)$$

and by $(L_{01} \times L_{12})^T \subset M_{0211}$ the transposed Lagrangian gotten by permuting the factors of M_{0211} by $(x_0, x_1, y_1, x_2) \mapsto (x_0, x_2, x_1, y_1)$.

The notion of "symmetric complex structure" in the following theorem will be defined in §2.2.1.

Theorem 2.2.1. *There exists $\epsilon > 0$ such that the following holds: Fix $k \in \mathbb{N}_{\geq 1}$, positive reals $\delta^\nu \to 0$ and $\rho > 0$, symmetric complex structures j^ν on $[-\rho,\rho]^2$ that converge $\mathcal{C}^\infty$ to j^∞ with $\|j^\infty - i\|_{\mathcal{C}^0} \leq \epsilon$, and $\mathcal{C}^{k+2}_{\mathrm{loc}}$-bounded sequences of domain-dependent compatible almost complex structures $J^\nu_\ell : [-\rho,\rho]^2 \to \mathcal{J}_\ell(M_\ell,\omega_\ell)$, $\ell \in \{0,1,2\}$ such that the $\mathcal{C}^{k+1}$-limit of each (J^ν_ℓ) is a compatible $\mathcal{C}^\infty$ almost complex structures $J^\infty_\ell \colon [-\rho,\rho]^2 \to \mathcal{J}(M_\ell,\omega_\ell)$.*

Then if $(v^\nu_0, v^\nu_1, v^\nu_2)$ is a sequence of size-(δ^ν,ρ) $(J^\nu_0, J^\nu_1, J^\nu_2, j^\nu)$-holomorphic squiggly strip quilts for (L_{01}, L_{12}) with uniformly bounded gradients,

$$\sup_{\nu\in\mathbb{N},\,(s,t)\in[-\rho,\rho]^2} |\mathrm{d}v^\nu|(s,t) < \infty,$$

then there is a subsequence in which $(v^\nu_0(t-\delta^\nu))$, $(v^\nu_1|_{t=0})$, $(v^\nu_2(t+\delta^\nu))$ converge $\mathcal{C}^k_{\mathrm{loc}}$ to a $(J^\infty_0, J^\infty_2, i)$-holomorphic size-$\rho$ degenerate strip quilt $(v^\infty_0, v^\infty_1, v^\infty_2)$ for $L_{01} \times_{M_1} L_{12}$.

If the inequality $\liminf_{\nu\to\infty,(s,t)\in[-\rho,\rho]^2} |\mathrm{d}v^\nu|(s,t) > 0$ holds, then v^∞_0, v^∞_2 are not both constant.

The analysis in our proof of Theorem 2.2.1 will be phrased in terms of pairs of smooth maps $(w_{02}, \widehat{w}) = ((w_0, w_2), (w'_0, w'_2, w'_1, w_1))$:

$$\begin{gathered} w_{02} \colon (-\rho,\rho) \times [0, \rho - 2\delta) \to M_{02}, \qquad \widehat{w} \colon (-\rho,\rho) \times [0,\delta] \to M_{0211}, \\ (w_{02}, \widehat{w})(s,0) \in \Delta_{M_{02}} \times \Delta_{M_1}, \qquad \widehat{w}(s,\delta) \in (L_{01} \times L_{12})^T \qquad \forall\, s \in (-\rho,\rho), \end{gathered} \tag{2.21}$$

where δ is nonnegative. From now on we denote the domains of w_{02} and $\widehat{w}$ by

$$Q_{02,\delta,\rho} := (-\rho,\rho) \times [0, \rho - 2\delta), \qquad \widehat{Q}_{\delta,\rho} := (-\rho,\rho) \times [0,\delta],$$

and combine them into the notation $\mathbf{Q}_{\delta,\rho} := (Q_{02,\delta,\rho}, \widehat{Q}_{\delta,\rho})$. We denote the closures in $\mathbb{R}^2$ by

$$\overline{Q}_{02,\delta,\rho} := [-\rho,\rho] \times [0, \rho - 2\delta], \qquad \overline{\widehat{Q}}_{\delta,\rho} := [-\rho,\rho] \times [0,\delta].$$

For $\delta > 0, \rho > 0$ (resp. $\delta = 0, \rho > 0$), the setup $(2.21)_{\delta,\rho}$ is equivalent to a triple of smooth maps (v_0, v_1, v_2) with the same domain and targets as a size-(δ,ρ) squiggly strip quilt for (L_{01}, L_{12}) $(2.8)_{f=\delta}$ (resp. as a size-ρ degenerate strip quilt for $L_{01} \times_{M_1} L_{12}$ (2.11)) and that fulfill the seam conditions $(2.9)_{f=\delta}$ (resp. (2.12)) but are not necessarily pseudoholomorphic or of finite energy. Indeed, given such (v_0, v_1, v_2), define $(w_{02}, \widehat{w})$ like so:

$$\begin{gathered} w_{02}(s,t) := (v_0(s, -t - 2\delta), v_2(s, t + 2\delta)), \\ \widehat{w}(s,t) := (v_0(t - 2\delta), v_2(s, -t + 2\delta), v_1(s,-t), v_1(s,t)). \end{gathered} \tag{2.22}$$

Conversely, for $\delta \geq 0$ and $(w_{02}, \widehat{w})$ satisfying $(2.21)_{\delta,\rho}$, define (v_0, v_1, v_2) satisfying $(2.8)_{f=\delta}$, $(2.9)_{f=\delta}$ (for $\delta > 0$) or (2.11), (2.12) (for $\delta = 0$) like so:

$$\begin{gathered} v_0(s,t) := \begin{cases} w'_0(s, t + 2\delta), & -2\delta \leq t \leq -\delta, \\ w_0(s, -t - 2\delta), & t \leq -2\delta, \end{cases} \qquad v_2(s,t) := \begin{cases} w'_2(s, -t + 2\delta), & \delta \leq t \leq 2\delta, \\ w_2(s, t - 2\delta), & 2\delta \leq t, \end{cases} \\ v_1(s,t) := \begin{cases} w'_1(s,-t), & -\delta \leq t \leq 0, \\ w_1(s,t), & 0 \leq t \leq \delta. \end{cases} \end{gathered} \tag{2.23}$$

The transformations (2.22), (2.23) are inverse to one another.

The following proof of Theorem 2.2.1 uses several notions that will be defined in §2.2.1–2.2.2.

Proof of Theorem 2.2.1. We divide the proof into steps: in Step 1, we show that the squiggly strip quilts converge $\mathcal{C}^0_{\mathrm{loc}}$ in a subsequence. In Step 2, we upgrade this convergence to $\mathcal{C}^k_{\mathrm{loc}}$. Finally, we prove in Step 3 that if the gradient satisfies a lower bound at a sequence of points with limit on the boundary, then at least one of v_0^∞, v_2^∞ is nonconstant. Throughout this proof, C_1 will be a constant that may change from line to line.

Step 1. *After passing to a subsequence, $(v_0^\nu(t-\delta^\nu))$, $(v_1^\nu|_{t=0})$, $(v_2^\nu(t+\delta^\nu))$ converge $\mathcal{C}^0_{\mathrm{loc}}$ to a $(J_0^\infty, J_2^\infty, i)$-holomorphic size-$\rho$ degenerate strip quilt $(v_0^\infty, v_1^\infty, v_2^\infty)$ for $L_{01} \times_{M_1} L_{12}$.*

The Arzelà–Ascoli theorem implies that there exist continuous maps

$$v_0^\infty\colon (-\rho,\rho)\times(-\rho,0]\to M_0, \qquad v_1^\infty\colon (-\rho,\rho)\to M_1, \qquad v_2^\infty\colon (-\rho,\rho)\times[0,\rho)\to M_2$$

such that after passing to a subsequence, $(v_0^\nu(s,t-\delta^\nu))$, $(v_1^\nu|_{t=0})$, $(v_2^\nu(s,t+\delta^\nu))$ converge $\mathcal{C}^0_{\mathrm{loc}}$ to v_0^∞, v_1^∞, v_2^∞. Standard compactness for pseudoholomorphic curves (e.g. [McSa, Theorem B.4.2]) implies that this convergence takes place in $\mathcal{C}^k_{\mathrm{loc}}$ on the interior (i.e. away from the line $t=0$); in particular, v_0^∞ resp. v_2^∞ are J_0^∞- resp. J_2^∞-holomorphic on the interior, hence $\mathcal{C}^\infty$ by [McSa, Theorem B.4.1]. In fact, we claim that v_0^∞ and v_2^∞ are $\mathcal{C}^\infty$ on their full domains, and that they satisfy a generalized Lagrangian boundary condition in $L_{01}\times_{M_1} L_{12}$ at $t=0$.

Denote by $\overline{v}$ the map

$$\overline{v} := (v_0^\infty(-,0), v_1^\infty(-), v_1^\infty(-), v_2^\infty(-,0))\colon (-\rho,\rho)\to M_0^-\times M_1\times M_1^-\times M_2.$$

To show that v_0^∞, v_2^∞ satisfy a generalized Lagrangian boundary condition in $L_{01}\circ L_{12}$, we will show that for any $s\in(-\rho,\rho)$, $\overline{v}(s)$ lies in $L_{01}\times_{M_1}L_{12}$. The containment $\overline{v}(s)\in M_0\times\Delta_{M_1}\times M_2$ is clear. To show the containment $\overline{v}(s)\in L_{01}\times L_{12}$, we will show that $(v_0^\infty(s,0), v_1^\infty(s))$ lies in L_{01}; the proof that $(v_1^\infty(s), v_2^\infty(s,0))$ lies in L_{12} is analogous. Since $(v_0^\nu(s,-\delta^\nu), v_1^\nu(s,-\delta^\nu))$ lies in L_{01}, and since $(v_1^\nu|_{t=0})$ converges $\mathcal{C}^0_{\mathrm{loc}}$ to v_1^∞, it suffices to show that the distances $d(v_1^\nu(s,-\delta^\nu), v_1^\nu(s,0))$ converge to zero. This follows from the uniform gradient bound on (v_1^ν) and the convergence of δ^ν to zero.

Let us show that v_0^∞ and v_2^∞ are $\mathcal{C}^\infty$. We have already concluded that these maps are $\mathcal{C}^\infty$ on the interior, so it only remains to show that they are $\mathcal{C}^\infty$ at the boundary points, w.l.o.g. at $(0,0)$. For that purpose we choose a neighborhood $U\subset L_{01}\times_{M_1}L_{12}$ of $\overline{v}(0)$ such that $\pi_{02}|_U\colon U\to M_{02}$ is a smooth embedding. Then $\pi_{02}(U)\subset M_{02}$ is a noncompact embedded Lagrangian, and since v_0^∞ and v_2^∞ are continuous we find $\epsilon>0$ such that $\overline{v}((-\epsilon,\epsilon))$ is contained in U. Hence $(v_0^\infty, v_2^\infty)((-\epsilon,\epsilon)\times\{0\})$ is contained in $\pi_{02}(U)$, so standard elliptic regularity (e.g. [McSa, Theorem B.4.1][1]) applied to the map $(v_0^\infty(s,-t), v_2^\infty(s,t))$ shows that v_0^∞ and v_2^∞ are $\mathcal{C}^\infty$ at $(0,0)$. Since $\pi_{02}|_U$ is a diffeomorphism onto its image, $\overline{v}$ is $\mathcal{C}^\infty$ at 0 and thus we have shown that $v_0^\infty, v_1^\infty, v_2^\infty$ are $\mathcal{C}^\infty$.

Step 2. *After passing to a further subsequence, the convergence of $(v_0^\nu(s,t-\delta^\nu))$, $(v_1^\nu|_{t=0})$, $(v_2^\nu(s,t+\delta^\nu))$ takes place in $\mathcal{C}^k_{\mathrm{loc}}$.*

[1] The hypothesis of [McSa] that the Lagrangian submanifold is closed can be removed.

In order to establish $\mathcal{C}^k_{\mathrm{loc}}$ convergence near $(-\rho,\rho)\times\{0\}$, we cannot rely on [McSa, Theorem B.4.2]. Rather, we will establish uniform Sobolev bounds for all three sequences of maps. The compact Sobolev embeddings $H^{k+2}\hookrightarrow\mathcal{C}^k$ resp. $H^{k+1}\hookrightarrow\mathcal{C}^k$ for two-dimensional resp. one-dimensional domains will then provide $\mathcal{C}^k_{\mathrm{loc}}$-convergent subsequences.

Set $\mathbf{J}^\nu$ resp. $\mathbf{j}^\nu$ to be the coherent pair of almost complex structures resp. coherent collection of complex structures resulting from the transformations (2.30) resp. (2.29) applied to $J_0^\nu, J_1^\nu, J_2^\nu$ resp. j^ν, and set $(w_{02}^\nu,\widehat{w}^\nu)$ to be the $(\mathbf{J}^\nu,\mathbf{j}^\nu)$-holomorphic size-$(\delta^\nu,\rho)$ folded strip quilt resulting from the transformation (2.22) applied to $(v_0^\nu,v_1^\nu,v_2^\nu)$. Then w_{02}^ν resp. $\widehat{w}^\nu|_{t=0}$ converge $\mathcal{C}^0_{\mathrm{loc}}$ to

$$u_{02}(s,t):=(v_0^\infty(s,-t),v_2^\infty(s,t))\quad\text{resp.}\quad\overline{u}(s,t):=(v_0^\infty(s,0),v_2^\infty(s,0),v_1^\infty(s),v_1^\infty(s)),$$

where we have used the assumed $\mathcal{C}^1$-bounds on (v_0^ν), (v_2^ν). Since (J_0^ν) resp. $(J_1^\nu|_{t=0})$ resp. (J_2^ν) converge $\mathcal{C}^{k+1}$ to J_0^∞ resp. J_1^∞ resp. J_2^∞, and since $(J_0^\nu),(J_1^\nu),(J_2^\nu)$ are $\mathcal{C}^{k+2}$-bounded, (J_{02}^ν) resp. $(\widehat{J}^\nu|_{t=0})$ converge $\mathcal{C}^{k+1}$ to J_{02}^∞ resp. $\widehat{J}^\infty$; since j^ν converges in $\mathcal{C}^\infty_{\mathrm{loc}}$ to $\mathbf{j}^\infty$, the components of $\mathbf{j}^\nu$ converge in $\mathcal{C}^\infty_{\mathrm{loc}}$ to a coherent collection $\mathbf{j}^\infty$ of complex structures $\mathbf{j}^\infty$.

Fix $\rho'\in(0,\rho)$ and choose $\rho>\rho_1>\rho_2>\cdots>\rho_{k+2}=\rho'$. Set u_{δ^ν} to be the restriction and extension to $\mathbf{Q}_{\delta^\nu,\rho_1}$ of u as defined in (2.33). Due to the $\mathcal{C}^0_{\mathrm{loc}}$-convergence of w_{02}^ν resp. $\widehat{w}^\nu|_{t=0}$ to u_{02} resp. $\overline{u}$ and the uniform $\mathcal{C}^1$-bounds on $\widehat{w}^\nu$, we can express w_{02}^ν resp. $\widehat{w}^\nu$ on Q_{02,δ,ρ_1} resp. $\widehat{Q}_{\delta^\nu,\delta,\rho_1}$ for sufficiently large ν in terms of the corrected exponential maps $e_{u_{02,\delta^\nu}}$ resp. $e_{\widehat{u}_{\delta^\nu}}$ and sections $(\zeta_{02}^\nu,\widehat{\zeta}^\nu)\in\Gamma^{k+1}_{u_{\delta^\nu}}$ as introduced in §2.2.2:

$$w_{02}^\nu=e_{u_{02,\delta^\nu}}(\zeta_{02}),\qquad\widehat{w}^\nu=e_{\widehat{u}_{\delta^\nu}}(\widehat{\zeta}).$$

The sections $\zeta_{02}^\nu,\widehat{\zeta}^\nu$ converge to zero in $\mathcal{C}^0$ as $\nu\to\infty$, are uniformly bounded in $\mathcal{C}^1$, and satisfy boundary conditions (2.34) in the linearizations of $(L_{01}\times L_{12})^T$ and $M_0\times\Delta_{M_1}\times M_2$.

Iteration claim. *We bound $\|\mathcal{D}^\nu_{\zeta^\nu}\zeta^\nu\|_{\widetilde{H}^l(\mathbf{Q}_{\delta^\nu,\rho_l})}$ and $\|\zeta^\nu\|_{\widetilde{H}^l(\mathbf{Q}_{\delta^\nu,\rho_l})}$ for $l\in[1,k+2]$ by induction on l, where $\widetilde{H}^l$ and $\mathcal{D}^\nu$ are the modified Sobolev space and the linear delbar operator defined in §2.2.2 using $\mathbf{J}^\nu$, $\mathbf{j}^\nu$, and the pair of connections $\nabla=(\nabla_{02},\widehat{\nabla})$ constructed in Lemma 2.2.4.*

The first key fact for this claim is the formula

$$\mathcal{D}^\nu_{\zeta^\nu}\zeta^\nu=\mathrm{d}e_{u_{\delta^\nu}}(\zeta^\nu)^{-1}\big(\overline{\partial}_{\mathbf{J}^\nu,\mathbf{j}^\nu}(e_{u_{\delta^\nu}}(\zeta^\nu))-F^\nu(\zeta^\nu)\big)=:G^\nu(\zeta^\nu),\tag{2.24}$$

justified in (2.37), where $\overline{\partial}_{\mathbf{J}^\nu,\mathbf{j}^\nu}$ is the nonlinear delbar operator defined in (2.28). The relevant fact here is that G^ν is a pair of smooth maps

$$G_{02}^\nu\colon u_{02,\delta^\nu}^*\mathrm{T}M_{02}\to u_{02,\delta^\nu}^*\mathrm{T}M_{02},\qquad\widehat{G}^\nu\colon\widehat{u}_{\delta^\nu}^*\mathrm{T}M_{0211}\to\widehat{u}_{\delta^\nu}^*\mathrm{T}M_{0211}$$

that preserve fibers but do not necessarily respect their linear structure. Furthermore, for any k, G^ν is uniformly bounded in $\mathcal{C}^k$. The second key fact is Lemma 2.2.8, which is a collection of δ-independent elliptic estimates.

Since ζ^ν is uniformly bounded in $\mathcal{C}^1$, $\|\zeta^\nu\|_{H^1(\mathbf{Q}_{\delta^\nu,\rho_1})}$ and

$$\|\mathcal{D}^\nu_{\zeta^\nu}\zeta^\nu\|_{H^1(\mathbf{Q}_{\delta^\nu,\rho_1})}=\|G^\nu(\zeta^\nu)\|_{H^1(\mathbf{Q}_{\delta^\nu,\rho_1})}$$

are uniformly bounded. This establishes the base case of the iteration.

Next, say that ζ^ν and $\mathcal{D}^\nu_{\zeta^\nu}\zeta^\nu$ are uniformly bounded in $\widetilde{H}^l(\mathbf{Q}_{\delta^\nu,\rho_l})$ for some $l \in [1, k+1]$. Lemma 2.2.8 yields:

$$\|\zeta^\nu\|_{\widetilde{H}^{l+1}(\mathbf{Q}_{\delta^\nu,\rho_{l+1}})} \leq C_1\big(\|\mathcal{D}^\nu_{\zeta^\nu}\zeta^\nu\|_{\widetilde{H}^l(\mathbf{Q}_{\delta^\nu,\rho_l})} + \|\zeta^\nu\|_{H^0(\mathbf{Q}_{\delta^\nu,\rho_l})}\big). \tag{2.25}$$

It remains to bound $\|\mathcal{D}^\nu_{\zeta^\nu}\zeta^\nu\|_{\widetilde{H}^{l+1}(\mathbf{Q}_{\delta^\nu,\rho_{l+1}})}$. Since ζ^ν is uniformly bounded in $\widetilde{H}^{l+1}(\mathbf{Q}_{\delta^\nu,\rho_{l+1}})$, it is uniformly bounded in $\mathcal{C}^{l-1}(\mathbf{Q}_{\delta^\nu,\rho_{l+1}})$ by Lemma 2.2.9, which allows us to bound $\|\mathcal{D}^\nu_{\zeta^\nu}\zeta^\nu\|_{\widetilde{H}^{l+1}(\mathbf{Q}_{\delta^\nu,\rho_{l+1}})}$:

$$\begin{aligned}
\|\mathcal{D}^\nu_{\zeta^\nu}\zeta^\nu\|_{\widetilde{H}^{l+1}(\mathbf{Q}_{\delta^\nu,\rho_{l+1}})} \overset{(2.24)}{\leq} C_1\Big(&\sum_{\substack{\lambda_1,\ldots,\lambda_m\geq 1,\\ \lambda_1+\cdots+\lambda_m\leq l+1}} \big\||\nabla^{\lambda_1}\zeta^\nu|\cdots|\nabla^{\lambda_m}\zeta^\nu|\big\|_{H^0(\mathbf{Q}_{\delta^\nu,\rho_{l+1}})} \\
&+ \sum_{\substack{\lambda_1,\ldots,\lambda_m\geq 1,\\ \lambda_1+\cdots+\lambda_m}} \big\||\nabla^{\lambda_1}\zeta^\nu|\cdots|\nabla^{\lambda_m}\zeta^\nu|\big\|_{\mathcal{C}^0H^0(\mathbf{Q}_{\delta^\nu,\rho_{l+1}})} \\
&+ \sum_{\substack{\lambda_1\geq 0,\lambda_2,\ldots,\lambda_m\geq 1,\\ \lambda_1+\cdots+\lambda_m\leq l-1}} \big\||\nabla_s\nabla^{\lambda_1}\zeta^\nu||\nabla^{\lambda_2}\zeta^\nu|\cdots|\nabla^{\lambda_m}\zeta^\nu|\big\|_{\mathcal{C}^0H^0(\mathbf{Q}_{\delta^\nu,\rho_{l+1}})}\Big) \\
\leq C_1\Big(&\|\zeta^\nu\|_{H^{l+1}(\mathbf{Q}_{\delta^\nu,\rho_{l+1}})} + \sum_{m=0}^{l-1}\|\nabla^m\zeta^\nu\|_{\mathcal{C}^0H^1(\mathbf{Q}_{\delta^\nu,\rho_{l+1}})} + 1\Big) \\
\leq C_1\big(&\|\zeta^\nu\|_{\widetilde{H}^{l+1}(\mathbf{Q}_{\delta^\nu,\rho_{l+1}})} + 1\big).
\end{aligned}$$

This, together with (2.25), establishes the iteration step and completes the Iteration Claim.

The uniform bounds on $\|\zeta^\nu\|_{\widetilde{H}^{k+2}(\mathbf{Q}_{\delta^\nu,\rho_{k+2}})}$ and the $\mathcal{C}^k$-bounds that result from Lemma 2.2.9 yield uniform bounds on $\|w^\nu_{02}\|_{H^{k+2}(\mathbf{Q}_{\delta^\nu,\rho_{k+2}})}$, $\|\widehat{w}^\nu\|_{H^{k+2}(\mathbf{Q}_{\delta^\nu,\rho_{k+2}})}$, and $\|\widehat{w}^\nu|_{t=0}\|_{H^{k+1}((-\rho_{k+2},\rho_{k+2}))}$. These bounds induce uniform bounds on the H^{k+2}-norms of v^ν_0, v^ν_2 on the relevant subdomains of $(-\rho_{k+2},\rho_{k+2})^2$ and on the H^{k+1}-norms of $v^\nu_1|_{(-\rho_{k+2},\rho_{k+2})\times\{0\}}$. The compact embeddings $H^{k+2} \hookrightarrow \mathcal{C}^k$ resp. $H^{k+1} \hookrightarrow \mathcal{C}^k$ for two-dimensional resp. one-dimensional domains implies the desired $\mathcal{C}^k_{\text{loc}}$-convergence of $(v^\nu_0(s,t-\delta^\nu))$ resp. $(v^\nu_1(s,0))$ resp. $(v^\nu_2(s,t+\delta^\nu))$ to v^∞_0 resp. v^∞_1 resp. v^∞_2.

Step 3. *We show that if for some $\ell \in \{0,1,2\}$ and $\kappa > 0$ the gradient satisfies a lower bound $|\mathrm{d}v^\nu_\ell(0,\tau^\nu)| \geq \kappa$ for some $\tau^\nu \to \tau^\infty \in (-\rho,\rho)$, then at least one of v^∞_0, v^∞_2 is nonconstant.*

In the notation of Step 2, it suffices to show that if for some $\tau^\nu \to \tau^\infty \in [0,\rho)$ and $\kappa > 0$ the inequality $|\mathrm{d}w^\nu(0,\tau^\nu)| := |\mathrm{d}w^\nu_{02}(0,\tau^\nu)| + |\mathrm{d}\widehat{w}^\nu(0,\tau^\nu)| \geq \kappa$ is satisfied, then u_{02} is not constant. We prove the contrapositive of this statement: assuming that u_{02} is constant, we will show that the quantities $\lim_{\nu\to\infty}\sup_{t\in[0,\rho)}|\mathrm{d}w^\nu_{02}(0,t)|$ and $\lim_{\nu\to\infty}\sup_{t\in[0,\delta^\nu]}|\mathrm{d}\widehat{w}^\nu(0,t)|$ are both zero.

Since the convergence of (w^ν_{02}) to u_{02} takes place in $\mathcal{C}^1_{\text{loc}}$, the quantity $\lim_{\nu\to\infty}\sup_{t\in[0,\rho)}|\mathrm{d}w^\nu_{02}(0,t)|$ is zero. To see that the quantity $\lim_{\nu\to\infty}\sup_{t\in[0,\delta^\nu]}|\mathrm{d}\widehat{w}^\nu(0,t)|$ is also zero, note that by the last paragraph of Step 1, the limit $\overline{u}$ of $(\widehat{w}^\nu)$ is also constant, which implies the formula $\mathrm{d}\widehat{w}^\nu = \mathrm{d}e_{\widehat{u}_{\delta^\nu}}(\widehat{\zeta}^\nu)(\nabla\widehat{\zeta}^\nu)$. It follows that to prove the equality $\lim_{\nu\to\infty}\sup_{t\in[0,\delta^\nu]}|\widehat{\nabla}\widehat{w}^\nu(0,t)| = 0$, it suffices to prove the equality $\lim_{\nu\to\infty}\sup_{t\in[0,\delta^\nu]}|\widehat{\nabla}\widehat{\zeta}^\nu(0,t)| = 0$. We can now estimate, using the Sobolev inequality $\|-\|_{\mathcal{C}^0} \leq C_1\|-\|_{H^1}$ for one-dimensional domains whose lengths are bounded away from zero:

$$\begin{aligned}
\limsup_{\nu\to\infty} \sup_{t\in[0,\delta^\nu]} |\widehat{\nabla}\widehat{\zeta}^\nu(0,t)| &\le \lim_{\nu\to\infty} \sup_{t\in[0,\delta^\nu]} |\widehat{\nabla}\widehat{\zeta}^\nu(0,0)| + \lim_{\nu\to\infty} \sup_{t\in[0,\delta^\nu]} |\widehat{\nabla}\widehat{\zeta}(0,t) - \widehat{\nabla}\widehat{\zeta}(0,0)| \\
&= \lim_{\nu\to\infty} \sup_{t\in[0,\delta^\nu]} |\widehat{\nabla}\widehat{\zeta}^\nu(0,t) - \widehat{\nabla}\widehat{\zeta}^\nu(0,0)| \\
&\le \lim_{\nu\to\infty} C_1 \textstyle\int_0^{\delta^\nu} |\widehat{\nabla}_t\widehat{\nabla}\widehat{\zeta}^\nu(0,t)|\,\mathrm{d}t \\
&\le \lim_{\nu\to\infty} C_1(\delta^\nu)^{1/2} \left(\textstyle\int_0^{\delta^\nu} |\widehat{\nabla}_t\widehat{\nabla}\widehat{\zeta}^\nu(0,t)|^2\,\mathrm{d}t\right)^{1/2} \\
&\overset{\text{Sobolev}}{\le} \lim_{\nu\to\infty} C_1(\delta^\nu)^{1/2} \|\widehat{\zeta}\|_{H^3(\widehat{Q}_{\delta^\nu,\rho})} = 0.
\end{aligned}$$

This completes the contrapositive of Step 3, which concludes our proof of Theorem 2.2.1. □

2.2.1 Complex and almost complex structures in the folded and unfolded setups

Gromov Compactness Theorem 3.3.1 is proved by "straightening" the seams of a squiggly strip quilt. Pushing forward the standard complex structure from the squiggly strip quilt to the new quilt with horizontal seams produces a nonstandard complex structure, which is symmetric under conjugation. We axiomatize this property in the following definition.

Definition 2.2.2. Fix $\rho > 0$. A **symmetric complex structure on** $[-\rho,\rho]^2$ is a complex structure j such that the equality

$$j(s,t) = -\sigma \circ j(s,-t) \circ \sigma$$

holds for any $(s,t) \in [-\rho,\rho]^2$, where σ is the conjugation $\alpha\partial_s + \beta\partial_t \mapsto \alpha\partial_s - \beta\partial_t$.

When a symmetric complex structure, almost complex structures, and a pseudoholomorphic squiggly strip quilt are "pushed forward" by the folding operation (2.22), the result is a "coherent system of complex structures", a "coherent pair of almost complex structures", and a "pseudoholomorphic folded strip quilt", defined as follows.

Definition 2.2.3. Fix $\delta > 0$ and $\rho > 0$.

- **A coherent collection of complex structures j on** $\overline{\mathbf{Q}}_{\delta,\rho}$ is a pair $\mathbf{j} = (j_{02}, \widehat{j}) = ((j_0, j_2), (j'_0, j'_2, j'_1, j_1))$, where j_0, j_2 (resp. j'_0, j'_2, j'_1, j_1) are complex structures on $\overline{Q}_{02,\delta,\rho}$ (resp. on $\overline{\widehat{Q}}_{\delta,\rho}$) such that the following equalities hold for all $s \in (-\rho,\rho)$:

$$j_\ell(s,0) = -\sigma \circ j'_\ell(s,0) \circ \sigma, \tag{2.26}$$

$$j'_0(s,\delta) = j'_2(s,\delta), \qquad j'_1(s,\delta) = j_1(s,\delta), \qquad j'_1(s,\delta) = -\sigma \circ j'_0(s,\delta) \circ \sigma. \tag{2.27}$$

- **A coherent pair of almost complex structures J on** $\overline{\mathbf{Q}}_{\delta,\rho}$ is a pair $\mathbf{J} = (J_{02}, \widehat{J})$, where $J_{02}, \widehat{J}$ are almost complex structures

$$J_{02} : \overline{Q}_{02,\delta,\rho} \to \mathcal{J}(M_{02}, \omega_{02}), \qquad \widehat{J} : \overline{\widehat{Q}}_{\delta,\rho} \to \mathcal{J}(M_{0211}, \omega_{0211})$$

satisfying the following compatibility condition: Set $\iota\colon M_{02} \to M_{0211}$ resp. $\pi\colon M_{0211} \to M_{02}$ to be the inclusion resp. projection. Then for any $s \in (-\rho,\rho)$, the following equality must hold:

$$J_{02}(s,0) = -\mathrm{d}\pi \circ \widehat{J}(s,0) \circ \mathrm{d}\iota.$$

- Fix a coherent collection $\mathbf{j}$ of complex structures and a coherent pair $\mathbf{J}$ of almost complex structures on $\overline{\mathbf{Q}}_{\delta,\rho}$. A **$(\mathbf{J},\mathbf{j})$-holomorphic size-$(\delta,\rho)$ folded strip quilt** is a collection of smooth maps $w = (w_{02}, \widehat{w}) = ((w_0, w_2), (w_0', w_2', w_1', w_1))$ satisfying (2.21) that have finite energy,

$$\int_{Q_{02,\delta,\rho}} u_{02}^* \omega_{02} < \infty, \qquad \int_{\widehat{Q}_{\delta,\rho}} \widehat{u}^* \omega_{0211} < \infty,$$

  and satisfy the Cauchy–Riemann equations

$$\overline{\partial}_{\mathbf{J},\mathbf{j}} w = (\overline{\partial}_{02,J_{02},j_{02}} w_{02}, \widehat{\overline{\partial}}_{\widehat{J},\widehat{j}} \widehat{w}) = 0,$$

  where $\overline{\partial}_{\mathbf{J},\mathbf{j}} = (\overline{\partial}_{02,J_{02},j_{02}}, \widehat{\overline{\partial}}_{\widehat{J},\widehat{j}})$ is the pair of operators defined by:

$$\begin{aligned} \overline{\partial}_{02,J_{02},j_{02}} w_{02} &:= (\mathrm{d}w_0, \mathrm{d}w_2) \circ (j_0, j_2)(\partial_s) - J_{02}(-, w_{02}) \circ (\partial_s w_0, \partial_s w_2), \\ \widehat{\overline{\partial}}_{\widehat{J},\widehat{j}} \widehat{w} &:= (\mathrm{d}w_0', \mathrm{d}w_2', \mathrm{d}w_1', \mathrm{d}w_1) \circ (j_0', j_2', j_1', j_1)(\partial_s) - \widehat{J}(-, \widehat{w}) \circ (\partial_s w_0', \partial_s w_2', \partial_s w_1', \partial_s w_1). \end{aligned} \tag{2.28}$$

Given a (J_0, J_1, J_2, j)-holomorphic squiggly strip quilt (v_0, v_1, v_2) with j symmetric, we can produce a folded strip quilt like this: Define a coherent collection $\mathbf{j}$ of complex structures by

$$\begin{aligned} j_{02}(s,t) = (j_0, j_2)(s,t) &:= (-\sigma \circ j(s, -t-2\delta) \circ \sigma, j(s, t+2\delta)), \qquad (2.29) \\ \widehat{j}(s,t) = (j_0', j_2', j_1', j_1)(s,t) &:= (j(s, t-2\delta), -\sigma \circ j(s, -t+2\delta) \circ \sigma, -\sigma \circ j(s,-t) \circ \sigma, j(s,t)) \end{aligned}$$

and a coherent pair $\mathbf{J}$ of almost complex structures by

$$\begin{aligned} J_{02}(s,t) &:= (-J_0(s, -t-2\delta)) \oplus J_2(s, t+2\delta), \\ \widehat{J} &:= J_0(t-2\delta) \oplus (-J_2(-t+2\delta)) \oplus (-J_1(s,-t)) \oplus J_1(s,t). \end{aligned} \tag{2.30}$$

If $(w_{02}, \widehat{w})$ is defined by applying (2.22) to (v_0, v_1, v_2), then $(w_{02}, \widehat{w})$ is a $(\mathbf{J},\mathbf{j})$-holomorphic size-(δ,ρ) folded strip quilt. Indeed, $(w_{02}, \widehat{w})$ have the correct domains and codomains and satisfy the seam conditions, as discussed earlier, and the finite-energy hypothesis on (v_0, v_1, v_2) implies that $(w_{02}, \widehat{w})$ has finite energy. The Cauchy–Riemann equation (2.10) for v_0 on $(-\rho, \rho) \times (-\rho, -2\delta]$ can be rewritten as

$$\mathrm{d}w_0(s,t) \circ (-\sigma \circ j(s, -t-2\delta) \circ \sigma) - (-J_0(s, -t-2\delta, w_0(s,t))) \circ \mathrm{d}w_0(s,t) = 0$$

for $w_0(s,t) := v_0(s, -t-2\delta)$ as in (2.22), so w_0 is $(-J_0(s,-t-2\delta), j_0(s,t))$-holomorphic on $Q_{02,\delta,\rho}$. Five similar calculations complete the check that $(w_{02}, \widehat{w})$ is $(\mathbf{J},\mathbf{j})$-holomorphic.

Finally, we consider the coordinate representation of a coherent collection of complex structures. Fix a coherent collection $\mathbf{j} = ((j_0, j_2), (j_0', j_2', j_1', j_1))$ of complex structures on $\overline{\mathbf{Q}}_{\delta,\rho}$. Define $a_0(s,t), c_0(s,t) \in \mathbb{R}$ by

$$j_0(s,t)(\partial_s) =: a_0(s,t)\partial_s + c_0(s,t)\partial_t, \tag{2.31}$$

and define $a_j(s,t), c_j(s,t)$ for $j \in \{1,2\}$ and $a'_k(s,t), c'_k(s,t)$ for $k \in \{0,1,2\}$ in the same way. Then (2.26) and (2.27) translate into the following conditions on these coefficients:

$$\begin{aligned} a_j(s,0) &= -a'_j(s,0), \quad & c_j(s,0) &= c'_j(s,0) \qquad & \forall\, j &\in \{0,1,2\}, \\ a_0(s,\delta) &= a_2(s,\delta), & a'_1(s,\delta) &= a_1(s,\delta), & a_0(s,\delta) &= -a'_1(s,\delta), \\ c_0(s,\delta) &= c_2(s,\delta), & c'_1(s,\delta) &= c_1(s,\delta), & c_0(s,\delta) &= c'_1(s,\delta). \end{aligned} \tag{2.32}$$

We will use this coordinate representation in §2.2.2.

2.2.2 A collection of strip-width–independent elliptic estimates

This subsection is devoted to proving Lemma 2.2.8, which is the crucial δ-independent elliptic estimate needed for the proof of Theorem 2.2.1.

In addition to the data fixed at the beginning of §2.2, fix for §2.2.2 $\rho > 0$ and a pair of maps $u = (u_{02}, \overline{u})$ satisfying $(2.21)_{\delta=0,\rho}$.

Furthermore, we continue to denote by $\mathbf{i}$ the standard coherent collection of complex structures, and for any $\delta \in (0, \rho/4]$ we define a pair $u_\delta = (u_{02,\delta}, \widehat{u}_\delta)$ of smooth maps satisfying $(2.21)_{\delta,\rho}$ by:

$$u_{02,\delta} := u_{02}|_{Q_{02,\delta,\rho}}, \qquad \widehat{u}_\delta(s,t) := \overline{u}(s). \tag{2.33}$$

Our approach is inspired by [WeWo4], but we deviate from that approach by working with a special connection which allows us to drop boundary terms from the H^2-estimate [WeWo4, Lemma 3.2.1(b)]. This special connection is constructed in the following lemma, which is a generalization to the immersed case of a connection constructed in [We5].

Lemma 2.2.4. *There is an assignment $\delta \mapsto \nabla_\delta = (\nabla_{02,\delta}, \widehat{\nabla}_\delta)$ that sends $\delta \in (0, \rho/4]$ to a pair of connections $\nabla_{02,\delta}$ resp. $\widehat{\nabla}_\delta$ on $u^*_{02,\delta}\mathrm{T}M_{02} \to Q_{02,\delta,\rho}$ resp. $\widehat{u}^*_\delta \mathrm{T}M_{0211} \to \widehat{Q}_{\delta,\rho}$ such that the following hold:*

- *Parallel transport under $\widehat{\nabla}_\delta$ preserves $\widehat{u}^*_\delta\mathrm{T}(L_{01} \times L_{12})^T$ and $\widehat{u}^*_\delta\mathrm{T}(M_{02} \times \Delta_{M_1})$;*
- *For a section $\widehat{\zeta} \in \Gamma(\widehat{u}^*_\delta\mathrm{T}(M_{02} \times \Delta_{M_1}))$ we have $\nabla_{02,\delta,s}(p \circ \widehat{\zeta}) = p \circ \widehat{\nabla}_{\delta,s}\widehat{\zeta}$, where $p\colon \widehat{u}^*_\delta\mathrm{T}(M_{02} \times \Delta_{M_1}) \to u^*_{02,\delta}\mathrm{T}M_{02}|_{t=0}$ is the projection;*
- *For $\delta_1 < \delta_2$, the restrictions of $\nabla_{\delta_1}, \nabla_{\delta_2}$ agree:*

$$\nabla_{02,\delta_1}|_{Q_{02,\delta_2,\rho}} = \nabla_{02,\delta_2}, \qquad \widehat{\nabla}_{\delta_2}|_{\widehat{Q}_{\delta_1,\rho}} = \widehat{\nabla}_{\delta_1}.$$

Proof. Fix metrics on $u^*_{02}\mathrm{T}M_{02}$ and $\overline{u}^*\mathrm{T}M_{0211}$ so that given a smooth subbundle, we may form its orthogonal complement. For any fixed $s \in (-\rho, \rho)$ we denote:

$$\Lambda_{0211} := \mathrm{T}_{\overline{u}(s)}(L_{01} \times L_{12})^T, \quad \Delta := \mathrm{T}_{\overline{u}(s)}(M_{02} \times \Delta_{M_1}),$$
$$\widehat{\Lambda}_{02} := \Lambda_{0211} \cap \Delta, \quad \Lambda_{02} := \mathrm{T}\pi_{02,\overline{u}(s)}(\widehat{\Lambda}_{02}).$$

The transversality of $L_{01} \times L_{12} \pitchfork M_0 \times \Delta_{M_1} \times M_2$ implies $\widehat{\Lambda}_{02} = \mathrm{T}_{\overline{u}(s)}\widehat{L}_{02}$, so the projection from $\widehat{\Lambda}_{02}$ to Λ_{02} is injective (see e.g. [WeWo4, Lemma 2.0.5]). Hence the intersection of $\widehat{\Lambda}_{02}$ and $\{0\} \times \mathrm{T}_{(\overline{u}_1(s),\overline{u}_1(s))}\Delta_{M_1}$ is trivial. It follows that if we let C_1 denote the complement of $\widehat{\Lambda}_{02} + (\{0\} \times \mathrm{T}_{(\overline{u}_1(s),\overline{u}_1(s))}\Delta_{M_1})$ in Δ, the diagonal decomposes as

$\Delta = \widehat{\Lambda}_{02} \oplus C_1 \oplus (\{0\} \times \mathrm{T}_{(\overline{u}_1(s),\overline{u}_1(s))}\Delta_{M_1})$. Let C_2 be the complement of $\widehat{\Lambda}_{02}$ in Λ_{0211}. Transversality implies $\mathrm{T}_{\overline{u}(s)}M_{0211} = \Lambda_{0211} + \Delta$, so we have deduced the following decomposition:

$$\mathrm{T}_{\overline{u}(s)}M_{0211} = C_2 \oplus \widehat{\Lambda}_{02} \oplus C_1 \oplus (\{0\} \times \mathrm{T}_{(\overline{u}_1(s),\overline{u}_1(s))}\Delta_{M_1}).$$

The subspace Λ_{0211} (resp. Δ) is given by the sum of the first two factors (resp. the sum of the last three factors) in this decomposition. Therefore, if we choose connections on each of these four subbundles and set $\overline{\nabla}$ to be the product connection, then extend $\overline{\nabla}$ to a connection $\widehat{\nabla}_\delta$ on $\widehat{u}_\delta^*\mathrm{T}M_{0211} \to \widehat{Q}_{\delta,\rho}$ by defining $\widehat{\nabla}_{\delta,s}((s,t) \mapsto \widehat{\zeta}(s,t)) := \overline{\nabla}_s(s \mapsto \widehat{\zeta}(s,t))$ and defining $\widehat{\nabla}_{\delta,t}((s,t) \mapsto \widehat{\zeta}(s,t)) := \nabla_{\widehat{g},t}(t \mapsto \widehat{\zeta}(s,t))$ in terms of the Levi-Civita connection $\nabla_{\widehat{g}}$, $\widehat{\nabla}_\delta$ satisfies the first bullet.

Denote by $p\colon \overline{u}^*\mathrm{T}(M_{02} \times \Delta_{M_1}) \to u_{02}^*\mathrm{T}M_{02}|_{t=0}$ projection and by $i\colon u_{02}^*\mathrm{T}M_{02}|_{t=0} \to u_{02}^*\mathrm{T}(M_{02} \times \Delta_{M_1})$ the inclusion defined by sending $v \in T_{u_{02}(s,0)}M_{02}$ to $(v,0) \in T_{\overline{u}(s)}(M_{02} \times \Delta_{M_1})$. Define a connection $p_*\overline{\nabla}$ on $u_{02}^*\mathrm{T}M_{02}|_{t=0}$ by $(p_*\overline{\nabla})(\zeta_{02}) := p \circ \overline{\nabla}(i \circ \zeta_{02})$. Extend $p_*\overline{\nabla}$ in any way to a connection ∇_{02} on $u_{02}^*\mathrm{T}M_{02}$; for $\delta \in (0, \rho/4]$, define $\nabla_{02,\delta} := \nabla_{02}|_{Q_{02,\delta,\rho}}$. The second bullet now follows from a computation, in which $(\zeta_{02}, \widehat{\zeta}_1, \widehat{\zeta}_1)$ is an arbitrary section of $\widehat{u}_\delta^*\mathrm{T}(M_{02} \times \Delta_{M_1})$:

$$p \circ \widehat{\nabla}_{\delta,s}\widehat{\zeta} \;=\; p \circ \widehat{\nabla}_{\delta,s}(\widehat{\zeta}_{02}, \widehat{\zeta}_1, \widehat{\zeta}_1) \;=\; p \circ \widehat{\nabla}_{\delta,s}(i \circ p \circ \widehat{\zeta}) + p \circ \widehat{\nabla}_{\delta,s}(0, \widehat{\zeta}_1, \widehat{\zeta}_1) \;=\nabla_{02,\delta,s}(p \circ \widehat{\zeta}).$$

The term $p\circ\widehat{\nabla}_{\delta,s}(0, \widehat{\zeta}_1, \widehat{\zeta}_1)$ in the third quantity vanishes since the subbundle $\{0\}\times\mathrm{T}_{(\widehat{w}_{\delta,1},\widehat{w}_{\delta,1})}\Delta_{M_1}$ is preserved under parallel transport by $\widehat{\nabla}_{\delta,s}$. □

We will use the connections ∇_δ just constructed throughout the rest of §2.2.2. Due to the third property in Lemma 2.2.4, it is unambiguous to drop the subscript and refer to ∇_δ simply as ∇. Note that this pair of connections induce connections on the pullbacks by $u_{02,\delta}$ or $\widehat{u}_\delta$ of any tensor bundle of $\mathrm{T}M_{02}$ or $\mathrm{T}M_{0211}$ in a canonical way.

Before we state the elliptic estimate Lemma 2.2.8, we need to define our function spaces and delbar operators.

Definition 2.2.5. Fix $r \in (0, \rho)$, $\delta > 0$, and $k \geq 2$. Define the space of sections $\mathbf{\Gamma^k_{u_\delta}(Q_{\delta,r})}$ and the norms $\| - \|_{\mathbf{H^k(Q_{\delta,r})}}, \| - \|_{\mathbf{\widetilde{H}^k(Q_{\delta,r})}}$ as follows.

- Define $\mathbf{\Gamma^k_{u_\delta}(Q_{\delta,r})}$ by:

$$\Gamma^k_{u_\delta}(\mathbf{Q}_{\delta,r}) := \left\{ \begin{pmatrix} \xi_{02} \in H^k(Q_{02,\delta,r}, u_{02,\delta}^*\mathrm{T}M_{02}), \\ \widehat{\xi} \in H^k(\widehat{Q}_{\delta,r}, \widehat{u}_\delta^*\mathrm{T}M_{0211}) \end{pmatrix} \middle| \; (2.34) \right\},$$

where (2.34) denotes the following linearized boundary conditions:

$$(\xi_{02}(s,0), \widehat{\xi}(s,0)) \in \mathrm{T}\Delta_{M_{02}} \times \mathrm{T}\Delta_{M_1}, \qquad \widehat{\xi}(s,\delta) \in \mathrm{T}(L_{01} \times L_{12})^T \qquad \forall\, s \in (-r,r). \tag{2.34}$$

- Define two norms $\|-\|_{\mathbf{H^k(Q_{\delta,r})}}, \|-\|_{\widetilde{\mathbf{H}}^{\mathbf{k}}(\mathbf{Q_{\delta,r}})}$ on $\Gamma^k_{u_\delta}$ by:

$$\|(\xi_{02},\widehat{\xi})\|^2_{H^k(\mathbf{Q}_{\delta,r})} := \|\xi_{02}\|^2_{H^k(Q_{02,r},u^*_{02,\delta}\mathrm{T}M_{02})} + \|\widehat{\xi}\|^2_{H^k(\widehat{Q}_{\delta,r},\widehat{u}^*_\delta\mathrm{T}M_{0211})},$$

$$\|(\xi_{02},\widehat{\xi})\|^2_{\widetilde{H}^k(\mathbf{Q}_{\delta,r})} := \|(\xi_{02},\widehat{\xi})\|^2_{H^k(\mathbf{Q}_{\delta,r})} + \sum_{l=0}^{k-2} \|(\nabla^l\xi_{02},\nabla^l\widehat{\xi})\|^2_{C^0H^1(\mathbf{Q}_{\delta,r})}$$

$$:= \|(\xi_{02},\widehat{\xi})\|^2_{H^k(\mathbf{Q}_{\delta,r})} + \sum_{l=0}^{k-2}\Big(\sup_{t\in[0,r-2\delta)} \|\nabla^l_{02}\xi_{02}(-,t)\|^2_{H^1((-r,r),u_{02,\delta}(-,t)^*\mathrm{T}M_{02})}$$
$$+ \sup_{t\in[0,\delta]} \|\widehat{\nabla}^l\widehat{\xi}(-,t)\|^2_{H^1((-r,r),\widehat{u}_\delta(-,t)^*\mathrm{T}M_{0211})}\Big).$$

Note that $\|-\|_{\widetilde{H}^k(\mathbf{Q}_{\delta,r})}$ is a well-defined norm on $\Gamma^k_{u_\delta}(\mathbf{Q}_{\delta,r})$ due to the embedding $H^1 \hookrightarrow C^0$ for one-dimensional domains. However, the constant in the bound $\|-\|_{\widetilde{H}^k(\mathbf{Q}_{\delta,r})} \leq C(\delta,r)\|-\|_{H^k(\mathbf{Q}_{\delta,r})}$ is δ-dependent.

In [WeWo4], Wehrheim–Woodward introduced an exponential map with quadratic corrections, which allowed them to treat the Lagrangian boundary conditions as totally geodesic. Wehrheim–Woodward assumed the composition $L_{01} \circ L_{12}$ to be embedded, but their construction of the corrected exponential map only used the immersedness of that composition. We may therefore import their corrected exponential map into our setting:

Definition 2.2.6. Given $r > 0$ and $\delta > 0$, define the **corrected exponential map $\mathbf{e}_{\mathbf{u}_\delta}$** and its **linearization $\mathrm{d}\mathbf{e}_{\mathbf{u}_\delta}$** and **s- and t-derivatives as follows.**

- Let $e_{u_\delta} = (e_{u_{02,\delta}}, e_{\widehat{u}_\delta})$ be the pair of maps defined in [WeWo4, Lemma 3.1.2]; e_{u_δ} sends $\zeta \in \Gamma^2_{u_\delta}(\mathbf{Q}_{\delta,r})$ with $\|\zeta\|_{C^0(\mathbf{Q}_{\delta,r})}$ sufficiently small to a pair of maps $e_{u_\delta}(\zeta) = (e_{u_{02,\delta}}(\zeta_{02}), e_{\widehat{u}_\delta}(\widehat{\zeta}))$ satisfying (2.21).

- For $p_{02} \in u^*_{02,\delta}\mathrm{T}M_{02}|_{(s,t)}$, $\mathrm{d}e_{u_{02,\delta}}(p_{02})\colon u^*_{02,\delta}\mathrm{T}M_{02}|_{(s,t)} \to \mathrm{T}_{e_{u_{02,\delta}}(p_{02})}M_{02}$ is defined by including the fiber $u^*_{02,\delta}\mathrm{T}M_{02}|_{(s,t)}$ into $\mathrm{T}_{p_{02}}u^*_{02,\delta}\mathrm{T}M_{02}$ as the vertical vectors, then postcomposing with the tangent map $\mathrm{T}(e_{u_{02,\delta}})_{p_{02}}\colon \mathrm{T}_{p_{02}}u^*_{02,\delta}\mathrm{T}M_{02} \to \mathrm{T}_{e_{u_{02,\delta}}(p_{02})}M_{02}$. The linearization $\mathrm{d}e_{\widehat{u}_\delta}(\widehat{p})$ is defined analogously.

- For $p_{02} \in u^*_{02,\delta}\mathrm{T}M_{02}|_{(s,t)}$, define $\mathbf{D_s e_{w_{02}}(p_{02})} \in \mathbf{T_{e_{w_{02}}(p_{02})}M_{02}}$ to be the vector gotten by choosing a flat section σ of $w^*_{02}\mathrm{T}M_{02}|_{(s-\epsilon,s+\epsilon)\times\{t\}}$ for ϵ small, then setting $\mathrm{D}_s e_{w_{02}}(p_{02}) := \mathrm{T}_s(e_w(\sigma))(\partial_s)$. The derivatives $\mathrm{D}_t e_{w_{02}}(p_{02}), \mathrm{D}_s e_{\widehat{w}}(\widehat{p}), \mathrm{D}_t e_{\widehat{w}}(\widehat{p})$ are defined analogously, and each of these derivatives depends smoothly on the argument p_{02} or $\widehat{p}$.

This exponential map will allow us to define fiberwise complex structures in the following, which are parametrized by vector fields rather than by maps.

In the following definition of the linear delbar operator, we must go into coordinates. Fix $\delta > 0$ and a coherent collection $\mathbf{j} = ((j_0,j_2),(j_0',j_2',j_1',j_1))$ of complex structures on $\overline{\mathbf{Q}}_{\delta,\rho}$. Then $\mathbf{j}$ induces via (2.31) two pairs of endomorphisms $A = (A_{02},\widehat{A})$, $C = (C_{02},\widehat{C})$ of $u^*_{02,\delta}\mathrm{T}M_{02}, \widehat{u}^*_\delta\mathrm{T}M_{0211}$, with $C_{02},\widehat{C}$ defined as follows and $A_{02},\widehat{A}$ defined in analogous fashion:

$$
\begin{aligned}
C_{02}(s,t)\colon \mathrm{T}_{u_{02,\delta}(s,t)}M_{02} &\to \mathrm{T}_{u_{02,\delta}(s,t)}M_{02}, \qquad (v_0,v_2)\mapsto (c_0(s,t)v_0, c_2(s,t)v_2), \qquad (2.35)\\
\widehat{C}(s,t)\colon \mathrm{T}_{\widehat{u}_\delta(s,t)}M_{0211} &\to \mathrm{T}_{\widehat{u}_\delta(s,t)}M_{0211},\\
&(v_0',v_2',v_1',v_1)\mapsto (c_0'(s,t)v_0', c_2'(s,t)v_2', c_1'(s,t)v_1', c_1(s,t)v_1).
\end{aligned}
$$

Note that the conditions (2.32) (which are equivalent to the coherence conditions (2.26), (2.27)) imply that for any $s\in[-\rho,\rho]$, the endomorphisms

$$
\widehat{C}(s,\delta), \qquad C_{02}(s,0)\times(\widehat{C}|_{(u_0',u_2')^*\mathrm{T}M_{02}})(s,0), \qquad (\widehat{C}|_{(u_1',u_1)^*\mathrm{T}M_{11}})(s,0)
$$

are scalar multiples of the identity; we will use this fact later in §2.2.2.

Definition 2.2.7. For $\delta>0$, $r>0$, $k\geq 2$, a coherent collection $\mathbf{j}$ of complex structures and a coherent pair of almost complex structures $\mathbf{J}$ on $\mathbf{Q}_{\delta,r}$, and $\xi\in\Gamma^2_{u_\delta}(\mathbf{Q}_{\delta,r})$, define the **linear delbar operator** $\mathcal{D}_\xi$ to be the following map from $H^1(Q_{02,\delta,r}, u_{02,\delta}^*\mathrm{T}M_{02})\times H^1(\widehat{Q}_{\delta,r},\widehat{u}_\delta^*\mathrm{T}M_{0211})$ to $H^0(u_{02,\delta}^*\mathrm{T}M_{02})\times H^0(\widehat{u}_\delta^*\mathrm{T}M_{0211})$:

$$
\begin{aligned}
\mathcal{D}_\xi\zeta &:= A\nabla_s\zeta + C\nabla_t\zeta - \mathbf{J}(\xi)\nabla_s\zeta\\
&:= \big(A_{02}\nabla_{02,s}\zeta_{02} + C_{02}\nabla_{02,t}\zeta_{02} - J_{02}(\xi_{02})\nabla_{02,s}\zeta_{02}, \widehat{A}\widehat{\nabla}_s\widehat{\zeta} + \widehat{C}\widehat{\nabla}_t\widehat{\zeta} - \widehat{J}(\widehat{\xi})\widehat{\nabla}_s\widehat{\zeta}\big),
\end{aligned}
$$

where $\mathbf{J}(\xi)$ is the **pulled-back complex structure**

$$
\begin{aligned}
\mathbf{J}(\xi)(s,t) &:= \mathrm{d}e_{u_\delta}(\xi(s,t))^{-1}\mathbf{J}(s,t,e_{u_\delta}(\xi(s,t)))\mathrm{d}e_{u_\delta}(\xi(s,t))\\
&:= \big(\mathrm{d}e_{u_{02,\delta}}(\xi_{02}(s,t))^{-1}J_{02}(s,t,e_{u_{02,\delta}}(\xi_{02}(s,t)))\mathrm{d}e_{u_{02,\delta}}(\xi_{02}(s,t)),\\
&\qquad \mathrm{d}e_{\widehat{u}_\delta}(\widehat{\xi}(s,t))^{-1}\widehat{J}(s,t,e_{\widehat{u}_\delta}(\widehat{\xi}(s,t)))\mathrm{d}e_{\widehat{u}_\delta}(\widehat{\xi}(s,t))\big).
\end{aligned}
$$

If $\zeta=(\zeta_{02},\widehat{\zeta})$ is a pair of sections in $\Gamma^2_{u_\delta}(\mathbf{Q}_{\delta,r})$, we can write $\partial_s(e_w(\zeta))$ and $\partial_t(e_w(\zeta)$ in terms of $\mathrm{d}e_{u_\delta}, \mathrm{D}_s e_{u_\delta}, \mathrm{D}_t e_{u_\delta}$:

$$
\begin{aligned}
\partial_s(e_{u_\delta}(\zeta)) &:= (\partial_s(e_{u_{02,\delta}}(\zeta_{02})), \partial_s(e_{\widehat{u}_\delta}(\widehat{\zeta}))) := (\mathrm{d}e_{u_{02,\delta}}(\zeta_{02})(\nabla_{02,s}\zeta_{02})\\
&\qquad + \mathrm{D}_s e_{u_{02,\delta}}(\zeta_{02}), \mathrm{d}e_{\widehat{u}_\delta}(\widehat{\zeta})(\widehat{\nabla}_s\widehat{\zeta}) + \mathrm{D}_s e_{\widehat{u}_\delta}(\widehat{\zeta})), \qquad (2.36)\\
\partial_t(e_{u_\delta}(\zeta)) &:= (\partial_t(e_{u_{02,\delta}}(\zeta_{02})), \partial_t(e_{\widehat{u}_\delta}(\widehat{\zeta}))) := (\mathrm{d}e_{u_{02,\delta}}(\zeta_{02})(\nabla_{02,t}\zeta_{02})\\
&\qquad + \mathrm{D}_t e_{u_{02,\delta}}(\zeta_{02}), \mathrm{d}e_{\widehat{u}_\delta}(\widehat{\zeta})(\widehat{\nabla}_t\widehat{\zeta}) + \mathrm{D}_t e_{\widehat{u}_\delta}(\widehat{\zeta})).
\end{aligned}
$$

This decomposition allows us to relate the delbar operator $\overline{\partial}_{\mathbf{J},\mathbf{j}}$ from (2.28) with the linear delbar operator $\mathcal{D}_\xi$ just defined:

$$
\begin{aligned}
\overline{\partial}_{\mathbf{J},\mathbf{j}}(e_{u_\delta}(\zeta)) &= A\partial_s(e_{u_\delta}(\zeta)) + C\partial_t(e_{u_\delta}(\zeta)) - \mathbf{J}(s,t,e_{u_\delta}(\zeta))\partial_s(e_{u_\delta}(\zeta))\\
&= \mathrm{d}e_{u_\delta}(\zeta)\big(A\nabla_s\zeta + C\nabla_t\zeta - \mathrm{d}e_{u_\delta}(\zeta)^{-1}\mathbf{J}(s,t,e_w(\zeta))\mathrm{d}e_{u_\delta}(\zeta)\nabla_s\zeta\big) \qquad (2.37)\\
&\qquad + \big(A\,\mathrm{D}_s e_{u_\delta}(\zeta) + C\,\mathrm{D}_t e_{u_\delta}(\zeta) - \mathbf{J}(s,t,e_{u_\delta}(\zeta))\mathrm{D}_s e_{u_\delta}(\zeta)\big)\\
&=: \mathrm{d}e_{u_\delta}(\zeta)\mathcal{D}_\zeta\zeta + F(\zeta).
\end{aligned}
$$

The inhomogeneous term F depends smoothly on ζ, which is crucial for the proof of Theorem 2.2.1.

The following is the main result of §2.2.2. It generalizes [WeWo4, Lemma 3.2.1], which bounds the H^1-norm of ζ when the domain complex structure is standard.

Lemma 2.2.8. *There is a constant $\epsilon > 0$ and for every $C_0 > 0$, $k \geq 0$, and r_1, r_2 with $0 < r_1 < r_2 < \rho$ there is a constant C_1 such that the inequality*

$$\|\zeta\|_{\widetilde{H}^{k+1}(\mathbf{Q}_{\delta,r_1})} \leq C_1\big(\|\mathcal{D}_\zeta\zeta\|_{\widetilde{H}^k(\mathbf{Q}_{\delta,r_2})} + \|\zeta\|_{H^0(\mathbf{Q}_{\delta,r_2})}\big) \tag{2.38}$$

holds for any choice of $\delta \in (0, r_1/4]$, a coherent collection $\mathbf{j}$ of complex structures on $\overline{\mathbf{Q}}_{\delta,\rho}$ with $\|\mathbf{j}-\mathbf{i}\|_{C^0} \leq \epsilon$ and $\|\mathbf{j}-\mathbf{i}\|_{C^{\max\{k,1\}}} \leq C_0$, a coherent pair $\mathbf{J}$ of almost complex structures on $\overline{\mathbf{Q}}_{\delta,\rho}$ which are contained in a $C^{\max\{k,1\}}$-ball of radius C_0 and which induce by (2.3) metrics whose pairwise constants of equivalence are bounded above by C_0, and a pair of sections $\zeta \in \Gamma^{k+2}_{u_\delta}(\mathbf{Q}_{\delta,r_2})$ with $\|\zeta\|_{C^0} \leq \epsilon$, $\|\zeta\|_{C^1} \leq C_0$, and $\|\zeta\|_{\widetilde{H}^k(\mathbf{Q}_{\delta,r_2})} \leq C_0$.

We begin by establishing δ-independent Sobolev estimates for elements of $\Gamma^k_{u_\delta}(\mathbf{Q}_{\delta,r})$.

Lemma 2.2.9. *Fix $C_0 > 0$, $k \geq 0$, and r_1, r_2 with $0 < r_1 < r_2 < \rho$. Then there is a constant C_1 and a polynomial P such that the inequality*

$$\begin{aligned}\|\nabla^k\zeta\|_{C^0H^1(\mathbf{Q}_{\delta,r})} \leq\ & C_1\big(\|\zeta\|_{H^{k+2}(\mathbf{Q}_{\delta,r})} + \|\nabla^{k-1}\mathcal{D}_\xi\zeta\|_{C^0H^1(\mathbf{Q}_{\delta,r})}\big)\\ &+ P\left(\sum_{l=1}^{k-1}\|\nabla^l\xi\|_{C^0H^1(\mathbf{Q}_{\delta,r})}\right)\left(\|\zeta\|_{H^{k+1}(\mathbf{Q}_{\delta,r})} + \sum_{l=0}^{k-2}\|\nabla^l\mathcal{D}_\xi\zeta\|_{C^0H^1(\mathbf{Q}_{\delta,r})}\right)\end{aligned} \tag{2.39}$$

(where the term $\|\nabla^{k-1}\mathcal{D}_\xi\zeta\|_{C^0H^1(\mathbf{Q}_{\delta,r})}$ is to be omitted when $k = 0$) holds for any choice of $\delta \in (0, r_1/4]$, $r \in [r_1, r_2]$, a coherent collection $\mathbf{j}$ of complex structures on $\overline{\mathbf{Q}}_{\delta,\rho}$ with $\|\mathbf{j}-\mathbf{i}\|_{C^k} \leq C_0$, a coherent pair $\mathbf{J}$ of compatible almost complex structures on $\mathbf{Q}_{\delta,\rho}$ which are contained in a C^k-ball of radius C_0 and which induce by (2.3) metrics whose pairwise constants of equivalence are bounded above by C_0, and pairs of sections $\zeta, \xi \in \Gamma^{k+2}_{u_\delta}(\mathbf{Q}_{\delta,r})$ with $\|\xi\|_{C^1} \leq C_0$.

Here is the idea of the proof: [WeWo4, Lemma 3.1.4] is a uniform Sobolev inequality for sections ζ satisfying the linearized boundary conditions. Since the special connection constructed in Lemma 2.2.4 preserves the linearized boundary conditions, [WeWo4, Lemma 3.1.4] immediately gives a bound on $\|\nabla^k_s\zeta\|_{C^0H^1(\mathbf{Q}_{\delta,r})}$. To derive a bound on $\|\nabla_\alpha\zeta\|_{C^0H^1(\mathbf{Q}_{\delta,r})}$ for $\alpha \in \{s,t\}^k$, we trade indices using the operator $\mathcal{D}_\xi$.

Proof. We prove this lemma in two steps: first, we prove a slightly different inequality, which has terms of the form $\|\nabla^l\zeta\|_{C^0H^1}$ on the right-hand side. Then, we prove the desired inequality by inductively removing these unwanted terms.

Throughout this proof, C_1 and P will denote a δ-independent constant and δ-independent polynomial that may change from line to line.

Step 1. *We prove the following inequality:*

$$\|\nabla^k\zeta\|_{C^0H^1} \leq C_1\left(\|\zeta\|_{H^{k+2}} + \|\nabla^{k-1}\mathcal{D}_\xi\zeta\|_{C^0H^1} + P\left(\sum_{l=1}^{k-1}\|\nabla^l\xi\|_{C^0H^1}\right)\cdot\sum_{l=0}^{k-1}\|\nabla^l\zeta\|_{C^0H^1}\right). \tag{2.40}$$

We begin by proving the $k=0$ case of (2.40), which is essentially a consequence of [WeWo4, Lemma 3.1.4]. One modification must be made to that lemma: we must relax the hypothesis that the composition $L_{01} \circ L_{12}$ is embedded to the hypothesis that this composition is immersed. To make this modification, change the proof of [WeWo4, Lemma 3.1.4] like so: instead of using [WeWo4, Lemma 3.1.3(c)], use the fact that for $\widehat{\xi} = (\xi'_{02}, \xi'_1, \xi_1) \in \mathcal{C}^\infty((-r,r), \overline{u}^*TM_{0211})$,

$$\|\widehat{\xi}\|_{H^1((-r,r))} \leq C_1\big(\|\xi'_{02}\|_{H^1((-r,r))} + \|\xi'_1 - \xi_1\|_{H^1((-r,r))} + \|\pi^\perp_{0211}\widehat{\xi}\|_{H^1((-r,r))}\big),$$

where $\pi^\perp_{0211}$ is the projection onto the orthogonal complement of the tangent space of $(L_{01} \times L_{12})^T$. This inequality follows from the pointwise estimate $|\widehat{\xi}| \leq C(|\xi'_{02}| + |\xi'_1 - \xi_1| + |\pi^\perp_{0211}\widehat{\xi}|)$, which can be proved like [WeWo4, Lemma 3.1.3b].

Next, fix $k \geq 1$; let us prove (2.40) for this k. Let ζ, ξ be sections in $\Gamma^{k+2}_{u_\delta}$, and assume that the other hypotheses of the lemma are satisfied. We will show that for every tuple $\alpha = (\alpha_1, \ldots, \alpha_k) \in \{s,t\}^k$, there is a polynomial P_α so that the following inequality holds:

$$\|\nabla_\alpha \zeta\|_{C^0H^1} \leq C_1\left(\|\zeta\|_{H^{k+2}} + \|\nabla^{k-1}\mathcal{D}_\xi\zeta\|_{C^0H^1} + P_\alpha\left(\sum_{l=1}^{k-1}\|\nabla^l\xi\|_{C^0H^1}\right)\cdot\sum_{l=0}^{k-1}\|\nabla^l\zeta\|_{C^0H^1}\right).$$

We prove this by induction on $n_t(\alpha) := \#\{m \in [1,k] \mid \alpha_m = t\}$.

$\boldsymbol{n_t(\alpha) = 0}$. If $\alpha = (s, \ldots, s)$, then since the special connection we have constructed preserves the boundary conditions of $\Gamma^{k+2}_{u_\delta}$, the desired inequality follows immediately from the $k=0$ case of the current lemma: $\|\nabla^k_s\zeta\|_{C^0H^1} \leq C_1\|\nabla^k_s\zeta\|_{H^2}$.

$\boldsymbol{n_t(\alpha) \in [1,k]}$. Let us prove the inductive step for some $n_t(\alpha) \in [1,k]$. Write $\alpha = (\alpha', \alpha_m = t, s, \ldots, s)$. Using the assumed bound on $\mathbf{j}$, we estimate:

$$\begin{aligned}\|\nabla_\alpha\zeta\|_{C^0H^1} &= \|\nabla_{\alpha'}(C^{-1}(\mathcal{D}_\xi(\nabla_s^{k-m}\zeta) - (A - \mathbf{J}(\xi))\nabla_s^{k-m+1}\zeta))\|_{C^0H^1}\\ &\leq C_1\Big(\|\nabla_{\alpha'}\mathcal{D}_\xi(\nabla_s^{k-m}\zeta)\|_{C^0H^1} + \|\nabla_{\alpha'}\nabla_s^{k-m+1}\zeta\|_{C^0H^1} + \|\nabla_{\alpha'}(\mathbf{J}(\xi)\nabla_s^{k-m+1}\zeta)\|_{C^0H^1}\\ &\qquad + \sum_{l=0}^{m-2}\|\nabla^{k-m+l+1}\zeta\|_{C^0H^1} + \sum_{l=0}^{m-2}\|\nabla^l(\mathbf{J}(\xi)\nabla^{k-m+1}\zeta)\|_{C^0H^1}\Big).\end{aligned}$$

Let us bound separately the five terms in the last expression.

$\boxed{\|\nabla_{\alpha'}\mathcal{D}_\xi(\nabla_s^{k-m}\zeta)\|_{C^0H^1}.}$ We estimate:

$$\begin{aligned}&\|\nabla_{\alpha'}\mathcal{D}_\xi(\nabla_s^{k-m}\zeta)\|_{C^0H^1}\\ &\leq \|\nabla_{\alpha'}\nabla_s^{k-m}\mathcal{D}_\xi\zeta\|_{C^0H^1} + \sum_{l=0}^{k-m-1}\|\nabla_{\alpha'}\nabla_s^l(\partial_sA\nabla_s^{k-m-l}\zeta + \partial_sC\nabla_s^{k-m-l-1}\nabla_t\zeta)\|_{C^0H^1}\\ &+ \sum_{l=1}^{k-m}\|\nabla_{\alpha'}\nabla_s^l(\mathbf{J}(\xi))\nabla_s^{k-m-l+1}\zeta\|_{C^0H^1} + \sum_{l=0}^{k-m-1}\|\nabla_{\alpha'}(C\nabla_s^l[\nabla_s,\nabla_t]\nabla_s^{k-m-l-1}\zeta)\|_{C^0H^1}.\end{aligned}$$

Let us bound each of the four terms on the right-hand side. The first term on the right-hand side, $\|\nabla_{\alpha'}\nabla_s^{k-m}\mathcal{D}_\xi\zeta\|_{C^0H^1}$, is bounded by $\|\nabla^{k-1}\mathcal{D}_\xi\zeta\|_{C^0H^1}$. Due to the assumed bound on $\mathbf{j}$, the term $\sum_{l=0}^{k-m-1}\|\nabla_{\alpha'}\nabla_s^l(\partial_sA\nabla_s^{k-m-l}\zeta + \partial_sC\nabla_s^{k-m-l-1}\nabla_t\zeta)\|_{C^0H^1}$

is bounded by a constant times $\sum_{l=0}^{k-1} \|\nabla^l \zeta\|_{C^0H^1}$. To bound the term

$$\sum_{l=1}^{k-m} \|\nabla_{\alpha'}\nabla_s^l(\mathbf{J}(\xi))\nabla_s^{k-m-l+1}\zeta\|_{C^0H^1},$$

observe that the assumed bound on $\mathbf{J}$ yields:

$$\begin{aligned}\sum_{l=1}^{k-m} \|\nabla_{\alpha'}\nabla_s^l(\mathbf{J}(\xi))\nabla_s^{k-m-l+1}\zeta\|_{C^0H^1} &\leq \sum_{\substack{\beta,\gamma\geq 0,\\ \beta+\gamma=k-2}} \|\nabla^{\beta+1}(\mathbf{J}(\xi))\nabla^{\gamma+1}\zeta\|_{C^0H^1} \\ &\leq P\left(\sum_{l=1}^{k-1} \|\nabla^l\xi\|_{C^0H^1}\right)\cdot\sum_{l=1}^{k-1}\|\nabla^l\zeta\|_{C^0H^1}.\end{aligned}$$

(In the last inequality we have used the Banach algebra property of C^0H^1.) Finally, the curvature of ∇ is a tensor, so the term $\sum_{l=0}^{k-m-1} \|\nabla_{\alpha'}(C\nabla_s^l[\nabla_s,\nabla_t]\nabla_s^{k-m-l-1}\zeta)\|_{C^0H^1}$ can be bounded by a constant times $\sum_{l=0}^{k-2}\|\nabla^l\zeta\|_{C^0H^1}$.

$\boxed{\|\nabla_{\alpha'}\nabla_s^{k-m+1}\zeta\|_{C^0H^1}.}$ By the inductive hypothesis, this term is bounded appropriately:

$$\|\nabla_{\alpha'}\nabla_s^{k-m+1}\zeta\|_{C^0H^1} \leq C_1\left(\|\zeta\|_{H^{k+2}} + \|\nabla^{k-1}\mathcal{D}_\xi\zeta\|_{C^0H^1} + P_{(\alpha',s,\ldots,s)}\left(\sum_{l=1}^{k-1}\|\nabla^l\xi\|_{C^0H^1}\right)\times \times\sum_{l=0}^{k-1}\|\nabla^l\zeta\|_{C^0H^1}\right).$$

$\boxed{\|\nabla_{\alpha'}(\mathbf{J}(\xi)\nabla_s^{k-m+1}\zeta)\|_{C^0H^1}.}$ To bound this term, it suffices to bound $\|\mathbf{J}(\xi)\nabla_{\alpha'}\nabla_s^{k-m+1}\zeta\|_{C^0H^1}$ and $\|\nabla^{\beta+1}(\mathbf{J}(\xi))\nabla^{\gamma+1}\zeta\|_{C^0H^1}$ separately, where in the second term β and γ are non-negative integers with $\beta+\gamma=k-2$. The quantity $\|\mathbf{J}(\xi)\nabla_{\alpha'}\nabla_s^{k-m+1}\zeta\|_{C^0H^1}$ can be bounded using the Banach algebra property of C^0H^1, the assumed C^1-bounds on ξ, and the inductive hypothesis. Using the Banach algebra property of C^0H^1, the quantity $\|\nabla^{\beta+1}(\mathbf{J}(\xi))\nabla^{\gamma+1}\zeta\|_{C^0H^1}$ can be bounded by $P\left(\sum_{l=1}^{k-1}\|\nabla^l\xi\|_{C^0H^1}\right)\cdot\sum_{l=1}^{k-1}\|\nabla^l\zeta\|_{C^0H^1}$.

$\boxed{\sum_{l=0}^{m-2}\|\nabla^{k-m+l+1}\zeta\|_{C^0H^1}.}$ This term is already bounded appropriately.

$\boxed{\sum_{l=0}^{m-2}\|\nabla^l(\mathbf{J}(\xi)\nabla^{k-m+1}\zeta)\|_{C^0H^1}.}$ By the Banach algebra property of C^0H^1, this term is bounded by $P\left(\sum_{l=1}^{k-2}\|\nabla^l\xi\|_{C^0H^1}\right)\cdot\sum_{l=1}^{k-1}\|\nabla^l\zeta\|_{C^0H^1}$.

This establishes the inductive step, so we have proven (2.40) for all $k\geq 0$.

Step 2. *We prove* (2.39) *by induction on* k.

As in Step 1, the $k=0$ case follows from [WeWo4, Lemma 3.1.4]. Next, say that (2.39) holds up to, but not including, some $k \geq 1$. By (2.40), we have:

$$\|\nabla^k \zeta\|_{C^0 H^1} \leq C_1 \left(\|\zeta\|_{H^{k+2}} + \|\nabla^{k-1} \mathcal{D}_\xi \zeta\|_{C^0 H^1} + P\left(\sum_{l=1}^{k-1} \|\nabla^l \xi\|_{C^0 H^1} \right) \cdot \sum_{l=0}^{k-1} \|\nabla^l \zeta\|_{C^0 H^1} \right).$$

Replacing the sum $\sum_{l=0}^{k-1} \|\nabla^l \zeta\|_{C^0 H^1}$ appearing in the last term using the inductive hypothesis finishes the inductive step. □

We now turn to the proof of Lemma 2.2.8. Here is our strategy: in Lemma 2.2.10, we bound $\|\zeta\|_{H^1}$ in terms of $\|\zeta\|_{H^0}$ and $\|\mathcal{D}_\zeta \zeta\|_{H^0}$, for ζ supported in $\mathbf{Q}_{\delta,r}$. In Lemma 2.2.11, we use Lemma 2.2.10 to bound $\|\eta \nabla^k \zeta\|_{H^1}$ in terms of $\|\zeta\|_{\widetilde{H}^k}$ and $\|\mathcal{D}_\zeta \zeta\|_{\widetilde{H}^k}$, where η is supported in $Q_{02,\delta,r}$ and ζ has arbitrary support. Finally, we use Lemma 2.2.11 to prove Lemma 2.2.8.

Lemma 2.2.10 (elliptic estimate for $k = 0$ and ζ compactly supported)**.** *There is a constant $\epsilon > 0$ and for every $C_0 > 0$, $k \geq 0$, and r_1, r_2 with $0 < r_1 < r_2 < \rho$ there is a constant C_1 such that the inequality*

$$\|\nabla \zeta\|_{H^0(\mathbf{Q}_{\delta,r})} \leq C_1 \big(\|\mathcal{D}_\xi \zeta\|_{H^0(\mathbf{Q}_{\delta,r})} + \|\zeta\|_{H^0(\mathbf{Q}_{\delta,r})} \big) \tag{2.41}$$

holds for any choice of $\delta \in (0, r_1/4]$, $r \in [r_1, r_2]$, a coherent collection $\mathbf{j}$ of complex structures on $\overline{\mathbf{Q}}_{\delta,\rho}$ with $\|\mathbf{j} - \mathbf{i}\|_{C^0} \leq \epsilon$ and $\|\mathbf{j} - \mathbf{i}\|_{C^1} \leq C_0$, a coherent pair $\mathbf{J}$ of almost complex structures on $\overline{\mathbf{Q}}_{\delta,\rho}$ which are contained in a C^1-ball of radius C_0 and which induce by (2.3) *metrics whose pairwise constants of equivalence are bounded above by C_0, and sections $\zeta, \xi \in \Gamma^2_{u_\delta}(\mathbf{Q}_{\delta,r})$ with $\|\xi\|_{C^0} \leq \epsilon$, $\|\xi\|_{C^1} \leq C_0$, and* $\operatorname{supp} \zeta_{02}$, $\operatorname{supp} \widehat{\zeta}$ *compact subsets of $Q_{02,\delta,r}$, $\widehat{Q}_{\delta,r}$.*

Proof. Throughout this proof, C_1 will denote a δ-independent constant that may change from line to line, and $A = (A_{02}, \widehat{A})$, $C = (C_{02}, \widehat{C})$ will be the endomorphisms of $u^*_{02,\delta} \mathrm{T} M_{02}$ and $\widehat{u}^*_\delta \mathrm{T} M_{0211}$ defined in (2.35).

We begin by fixing convenient metrics on M_{02} and M_{0211} that will be used for the pointwise norms in the definition of the Sobolev norms. Via (2.3), $\mathbf{J}$ induces fiberwise metrics $g_{02}, \widehat{g}$ on $u^*_{02,\delta} \mathrm{T} M_{02}$ and $\widehat{u}^*_\delta \mathrm{T} M_{0211}$. In this proof, however, we will use the pullback metrics $g_\xi = (g_{02,\xi}, \widehat{g}_\xi)$ of $g_{02}, \widehat{g}$ under $\mathrm{d}e_{u_{02,\delta}}(\xi_{02})$, $\mathrm{d}e_{\widehat{u}_\delta}(\widehat{\xi})$; note that g_ξ is $\mathbf{J}(\xi)$-invariant. If we pick $\epsilon > 0$ to be sufficiently small, then $\mathrm{d}e_{u_\delta}(\xi)$ is C^0-close to the identity, and hence the induced norms $\| - \|_{\xi, H^k}$ on $\Gamma^k_{u_\delta}(\mathbf{Q}_{\delta,r})$ are equivalent to the standard norms $\| - \|_{H^k} = \| - \|_{0,H^k}$.

With these metrics we calculate for $\zeta \in \Gamma^2_{u_\delta}$ compactly supported and $\xi \in \Gamma^2_{u_\delta}$ satisfying $\|\xi\|_{C^0(\mathbf{Q}_{\delta,r})} \leq \epsilon$ and $\|\nabla \xi\|_{C^0(\mathbf{Q}_{\delta,r})} \leq C_0$:

$$\begin{aligned} \|\mathcal{D}_\xi \zeta\|^2_{\xi,H^0} = \textstyle\int_{\mathbf{Q}_{\delta,r}} \big((|\nabla_s \zeta|^2_\xi + |A \nabla_s \zeta|^2_\xi) + 2 g_\xi(A \nabla_s \zeta, C \nabla_t \zeta) + |C \nabla_t \zeta|^2_\xi \big) \, \mathrm{d}s \mathrm{d}t \\ + \textstyle\int_{\mathbf{Q}_{\delta,r}} \big(g_\xi(C \nabla_s \zeta, \mathbf{J}(\xi) \nabla_t \zeta) - g_\xi(C \nabla_t \zeta, \mathbf{J}(\xi) \nabla_s \zeta) \big) \, \mathrm{d}s \mathrm{d}t. \end{aligned} \tag{2.42}$$

Let us estimate the two integrals on the right-hand side separately. We begin with the first integral:

$$\begin{aligned} &\textstyle\int_{\mathbf{Q}_{\delta,r}} \big((|\nabla_s \zeta|^2_\xi + |A \nabla_s \zeta|^2_\xi) + 2 g_\xi(2 A \nabla_s \zeta, \tfrac{1}{2} C \nabla_t \zeta) + |C \nabla_t \zeta|^2_\xi \big) \, \mathrm{d}s \mathrm{d}t \\ &\overset{\text{AM-GM}}{\geq} \textstyle\int_{\mathbf{Q}_{\delta,r}} \big((|\nabla_s \zeta|^2_\xi - 3|A \nabla_s \zeta|^2_\xi) + \tfrac{3}{4} |C \nabla_t \zeta|^2_\xi \big) \, \mathrm{d}s \mathrm{d}t \geq \tfrac{5}{8} \|\nabla \zeta\|^2_{\xi, H^0}, \end{aligned} \tag{2.43}$$

where the last inequality follows from the hypothesis $\|\mathbf{j}-\mathbf{i}\| \leq \epsilon$ as long as ϵ is chosen small enough.

To bound the second integral on the right-hand side of (2.42), we first derive a convenient formula for its integrand:

$$
\begin{aligned}
&g_\xi(C\nabla_s\zeta, \mathbf{J}(\xi)\nabla_t\zeta) - g_\xi(C\nabla_t\zeta, \mathbf{J}(\xi)\nabla_s\zeta) \qquad (2.44)\\
&\quad = \big(\partial_s(g_\xi(C\zeta, \mathbf{J}(\xi)\nabla_t\zeta)) - (\nabla_s g_\xi)(C\zeta, \mathbf{J}(\xi)\nabla_t\zeta) - g_\xi((\nabla_s C)\zeta, \mathbf{J}(\xi)\nabla_t\zeta)\\
&\qquad - g_\xi(C\zeta, \nabla_s(\mathbf{J}(\xi))\nabla_t\zeta) - g_\xi(C\zeta, \mathbf{J}(\xi)\nabla_s\nabla_t\xi)\big)\\
&\qquad - \big(\partial_t(g_\xi(C\zeta, \mathbf{J}(\xi)\nabla_s\zeta)) - \nabla_t(g_\xi)(C\zeta, \mathbf{J}(\xi)\nabla_s\zeta) - g_\xi((\nabla_t C)\zeta, \mathbf{J}(\xi)\nabla_s\zeta)\\
&\qquad - g_\xi(C\zeta, \nabla_t(\mathbf{J}(\xi))\nabla_s\zeta) + g_\xi(C\zeta, \mathbf{J}(\xi)[\nabla_s, \nabla_t]\zeta) - g_\xi(C\zeta, \mathbf{J}(\xi)\nabla_s\nabla_t\xi)\big)\\
&= \big(\partial_s(g_\xi(C\zeta, \mathbf{J}(\xi)\nabla_t\zeta)) - \partial_t(g_\xi(C\zeta, \mathbf{J}(\xi)\nabla_s\zeta))\big) - \big((\nabla_s g_\xi)(C\zeta, \mathbf{J}(\xi)\nabla_t\zeta) - (\nabla_t g_\xi)(C\zeta, \mathbf{J}(\xi)\nabla_s\zeta)\big)\\
&\quad - \big(g_\xi((\nabla_s C)\zeta, \mathbf{J}(\xi)\nabla_t\zeta) - g_\xi((\nabla_t C)\zeta, \mathbf{J}(\xi)\nabla_s\zeta)\big) - g_\xi(C\zeta, \nabla_s(\mathbf{J}(\xi))\nabla_t\zeta - \nabla_t(\mathbf{J}(\xi))\nabla_s\zeta)\\
&\qquad - g_\xi(C\zeta, \mathbf{J}(\xi)[\nabla_s, \nabla_t]\zeta).
\end{aligned}
$$

We can now use Green's formula and the assumed $\mathcal{C}^1$-bounds on $\mathbf{j}$, $\mathbf{J}$, and ξ to bound the second integral on the right-hand side of (2.43):

$$
\begin{aligned}
&\textstyle\int_{\mathbf{Q}_{\delta,r}} \big(g_\xi(C\nabla_s\zeta, \mathbf{J}(\xi)\nabla_t\zeta) - g_\xi(C\nabla_t\zeta, \mathbf{J}(\xi)\nabla_s\zeta)\big)\,\mathrm{d}s\mathrm{d}t\\
&\quad \overset{(2.44)}{=} \textstyle\int_{(-r,r)\times\{0\}} g_\xi(C\zeta, \mathbf{J}(\xi)\nabla_s\zeta)\,\mathrm{d}s\mathrm{d}t - \int_{(-r,r)\times\{\delta\}} \widehat{g}_\xi(\widehat{C}\widehat{\zeta}, \widehat{\mathbf{J}}(\widehat{\xi})\widehat{\nabla}_s\widehat{\zeta})\,\mathrm{d}s\mathrm{d}t \qquad (2.45)\\
&\qquad - \textstyle\int_{\mathbf{Q}_{\delta,r}} \big((\nabla_s g_\xi)(C\zeta, \mathbf{J}(\xi)\nabla_t\zeta) - (\nabla_t g_\xi)(C\zeta, \mathbf{J}(\xi)\nabla_s\zeta)\big)\,\mathrm{d}s\mathrm{d}t\\
&\qquad - \textstyle\int_{\mathbf{Q}_{\delta,r}} \big(g_\xi((\nabla_s C)\zeta, \mathbf{J}(\xi)\nabla_t\zeta) - g_\xi((\nabla_t C)\zeta, \mathbf{J}(\xi)\nabla_s\zeta)\big)\,\mathrm{d}s\mathrm{d}t\\
&\qquad - \textstyle\int_{\mathbf{Q}_{\delta,r}} g_\xi(C\zeta, \nabla_s\mathbf{J}(\xi))\nabla_t\zeta - \nabla_t(\mathbf{J}(\xi))\nabla_s\zeta)\,\mathrm{d}s\mathrm{d}t - \int_{\mathbf{Q}_{\delta,r}} g_\xi(C\zeta, \mathbf{J}(\xi)[\nabla_s, \nabla_t]\zeta)\,\mathrm{d}s\mathrm{d}t\\
&\quad \geq -\textstyle\int_{\mathbf{Q}_{\delta,r}} C_1|\zeta|_\xi(|\zeta|_\xi + |\nabla\zeta|_\xi)\,\mathrm{d}s\mathrm{d}t \overset{\text{AM-GM}}{\geq} -\frac{1}{2}\|\nabla\zeta\|^2_{\xi,H^0} - C_1\|\zeta\|^2_{\xi,H^0},
\end{aligned}
$$

where in the first inequality we have eliminated the integrals over the $t=0$ and $t=\delta$ boundary via the coherence condition on $\mathbf{j}$ and the fact that $g_\xi(\zeta, \mathbf{J}(\xi)\nabla_s\zeta)|_{t=0}$ and $\widehat{g}_\xi(\widehat{\zeta}, \widehat{J}(\widehat{\xi})\widehat{\nabla}_s\widehat{\zeta})|_{t=\delta}$ vanish. Indeed, $\widehat{g}_\xi(\widehat{\zeta}, \widehat{J}(\widehat{\xi})\widehat{\nabla}_s\widehat{\zeta})|_{t=\delta}$ vanishes by the Lagrangian boundary condition:

$$
\begin{aligned}
\langle\widehat{\zeta}, \widehat{J}(\widehat{\xi})\widehat{\nabla}_s\widehat{\zeta}\rangle_{\widehat{\xi}}|_{t=\delta} &= \omega_{0211}(\mathrm{d}e_{\widehat{u}_\delta}(\widehat{\xi})\widehat{\zeta}, \widehat{J}(e_{\widehat{u}_\delta}(\widehat{\xi}))^2\mathrm{d}e_{\widehat{u}_\delta}(\widehat{\xi})\widehat{\nabla}_s\widehat{\zeta})|_{t=\delta}\\
&= -\omega_{0211}(\mathrm{d}e_{\widehat{u}_\delta}(\widehat{\xi})\widehat{\zeta}, \mathrm{d}e_{\widehat{u}_\delta}(\widehat{\xi})\widehat{\nabla}_s\widehat{\zeta})|_{t=\delta} \;=\; 0,
\end{aligned}
$$

where we crucially used the fact that both the exponential map $\mathrm{d}e_{\widehat{u}_\delta}(\widehat{\xi})$ and the connection $\widehat{\nabla}$ preserve $\mathrm{T}(L_{01}\times L_{12})^T$. The boundary term $g_\xi(\zeta, \mathbf{J}(\xi)\nabla_s\zeta)|_{t=0}$ vanishes due to the facts that $\mathrm{d}e_{u_\delta}(\xi)$ preserves $\mathrm{T}\Delta_{M_{02}}\times\mathrm{T}\Delta_{M_1}$, ∇ satisfies $\nabla_{02,s}\zeta_{02}|_{t=0} = p\circ\widehat{\nabla}_s\widehat{\zeta}|_{t=0}$ for $p\colon M_{0211}\to M_{02}$ the projection, and $\omega_{02}, \omega_{0211}$ satisfy $\omega_{0211}|_{\mathrm{T}M_{02}\times\mathrm{T}\Delta_{M_1}} = -p^*\omega_{02}$:

$$
\begin{aligned}
\langle\zeta, \mathbf{J}(\xi)\nabla_s\zeta\rangle_\zeta|_{t=0} &= -\omega_{02}(\mathrm{d}e_{u_{02,\delta}}(\xi_{02})\zeta_{02}, \mathrm{d}e_{u_{02,\delta}}(\xi_{02})\nabla_{02,s}\zeta_{02})|_{t=0}\\
&\qquad - \omega_{0211}(\mathrm{d}e_{\widehat{u}_\delta}(\widehat{\xi})\widehat{\zeta}, \mathrm{d}e_{\widehat{u}_\delta}(\widehat{\xi})\widehat{\nabla}_s\widehat{\xi})|_{t=0}\\
&= -\omega_{02}(\mathrm{d}e_{u_{02,\delta}}(\xi_{02})(p\circ\widehat{\zeta}), \mathrm{d}e_{u_{02,\delta}}(\xi_{02})(p\circ\widehat{\nabla}_s\widehat{\zeta}))|_{t=0}\\
&\qquad + p^*\omega_{02}(\mathrm{d}e_{\widehat{u}_\delta}(\widehat{\xi})\widehat{\zeta}, \mathrm{d}e_{\widehat{u}_\delta}(\widehat{\xi})\widehat{\nabla}_s\widehat{\zeta})|_{t=0} = 0.
\end{aligned}
$$

Combining (2.42), (2.43), and (2.45) yields the following inequality:

$$\|\mathcal{D}_\xi\zeta\|^2_{\xi,H^0} \geq \tfrac{1}{8}\|\nabla\zeta\|^2_{\xi,H^0} - C_1\|\zeta\|^2_{\xi,H^0}.$$

Adding $C_1\|\zeta\|^2_{\xi,H^0}$ to both sides of this inequality and taking the square root of the result, we obtain:

$$\|\nabla\zeta\|_{\xi,H^0} \leq C_1(\|\mathcal{D}_\xi\zeta\|^2_{\xi,H^0} + \|\zeta\|^2_{\xi,H^0})^{1/2} \leq C_1(\|\mathcal{D}_\xi\zeta\|_{\xi,H^0} + \|\zeta\|_{\xi,H^0}).$$

In this estimate, we may replace $\| - \|_{\xi,H^0}$ with $\| - \|_{H^0}$ by using the δ-independent uniform equivalence of these norms, which yields (2.41). □

Lemma 2.2.11 (elliptic estimate for $k \geq 0$). *There is a constant $\epsilon > 0$ and for every $C_0 > 0$, $k \geq 0$, and $0 < r_1 < r_2 < \rho$ there is a constant C_1 such that the inequality*

$$\|\eta\nabla^k\zeta\|_{H^1(\mathbf{Q}_{\delta,r})} \leq C_1\big(\|\mathcal{D}_\zeta\zeta\|_{\widetilde{H}^k(\mathbf{Q}_{\delta,r})} + \|\zeta\|_{\widetilde{H}^k(\mathbf{Q}_{\delta,r})}\big) \tag{2.46}$$

holds for any choice of $\delta \in (0, r_1/4]$, $r \in [r_1, r_2]$, a coherent collection $\mathbf{j}$ of complex structures on $\overline{\mathbf{Q}}_{\delta,\rho}$ with $\|\mathbf{j}-\mathbf{i}\|_{C^0} \leq \epsilon$ and $\|\mathbf{j}-\mathbf{i}\|_{C^{\max\{k,1\}}} \leq C_0$, a pair $\mathbf{J}$ of compatible almost complex structures on $\overline{\mathbf{Q}}_{\delta,\rho}$ which are contained in a $C^{\max\{k,1\}}$-ball of radius C_0 and which induce by (2.3) metrics whose pairwise constants of equivalence are bounded above by C_0, a pair of sections $\zeta \in \Gamma^{k+2}_{u_\delta}(\mathbf{Q}_{\delta,r})$ with $\|\zeta\|_{C^0} \leq \epsilon$, $\|\zeta\|_{C^1} \leq C_0$, and $\|\zeta\|_{\widetilde{H}^k(\mathbf{Q}_{\delta,r})} \leq C_0$, and a smooth function $\eta\colon \overline{Q}_{02,\delta,r} \to \mathbb{R}$ with $\|\eta\|_{C^{k+1}} \leq c_0$ and $\operatorname{supp}\eta \subset Q_{02,\delta,r}$.

Proof. Throughout this proof, C_1 will denote a δ-independent constant and P will denote a δ-independent polynomial, and both may change from line to line.

We break down the proof into several steps: in Step 1, we establish (2.46), but with an extra term on the right-hand side. In Step 2, we bound this extra term, using different arguments in the $k \neq 3$ and $k = 3$ cases. In Step 3, we establish (2.46).

Step 1a. *We prove the following inequality:*

$$\|\eta\nabla_\alpha\zeta\|_{H^1} \leq C_1\Big(\|\mathcal{D}_\zeta\zeta\|_{H^k} + \|\zeta\|_{H^k} + \sum_{\substack{\beta\geq 1,\gamma\geq 0,\\ \beta+\gamma=k}} \|\eta\nabla^\beta(\mathbf{J}(\zeta))\nabla^\gamma\nabla_s\zeta\|_{H^0}\Big) \tag{2.47}$$

for $\alpha = \underbrace{(s,\ldots,s)}_{k}$.

Since the connection ∇ preserves the linearized boundary conditions and η is supported in

$Q_{02,\delta,r}$, we may estimate $\|\eta\nabla_s^k\zeta\|_{H^1}$ using Lemma 2.2.10:

$$\begin{aligned}
\|\eta\nabla_s^k\zeta\|_{H^1} &\leq C_1(\|\mathcal{D}_\zeta(\eta\nabla_s^k\zeta)\|_{H^0} + \|\eta\nabla_s^k\zeta\|_{H^0}) \\
&= C_1\Big(\|\eta\nabla_s^k\zeta\|_{H^0} + \Big\|\eta\nabla_s^k\mathcal{D}_\zeta\zeta - \sum_{l=1}^{k}\binom{k}{l}\eta(\partial_s^l A\nabla_s^{k-l+1}\zeta + \partial_s^l C\nabla_s^{k-l}\nabla_t\zeta) \\
&\qquad + \sum_{l=1}^{k}\binom{k}{l}\eta\nabla_s^l(\mathbf{J}(\zeta))\nabla_s^{k-l+1}\zeta - \sum_{l=1}^{k} C\eta\nabla_s^{l-1}[\nabla_s,\nabla_t]\nabla_s^{k-l}\zeta \\
&\qquad - (\partial_s\eta(A-\mathbf{J}(\zeta)) + C\partial_t\eta)\nabla_s^k\zeta\Big\|_{H^0}\Big) \\
&\leq C_1\Big(\|\mathcal{D}_\zeta\zeta\|_{H^k} + \|\zeta\|_{H^k} + \sum_{\substack{\beta\geq 1,\gamma\geq 0,\\ \beta+\gamma=k}}\|\eta\nabla^\beta(\mathbf{J}(\zeta))\nabla^\gamma\nabla_s\zeta\|_{H^0}\Big).
\end{aligned}$$

Step 1b. *We prove* (2.47) *for a general multiindex α of length k.*

We establish Step 1b by induction on $n_t(\alpha) := \{\#m \in [1,k] \mid \alpha_m = t\}$. Step 1a is the base case for this induction. For the inductive step, fix α with $n_t(\alpha) \geq 1$, and write $\alpha = (\alpha', \alpha_m = t, \underbrace{s,\ldots,s}_{k-m})$. We estimate:

$$\begin{aligned}
\|\eta\nabla_\alpha\zeta\|_{H^1} &= \|\eta\nabla_{\alpha'}(C^{-1}(\mathcal{D}_\zeta(\nabla_s^{k-m}\zeta) - (A-\mathbf{J}(\zeta))\nabla_s^{k-m+1}\zeta))\|_{H^1} \\
&\leq C_1(\|\zeta\|_{H^k} + \|\eta\nabla_{\alpha'}\mathcal{D}_\zeta(\nabla_s^{k-m}\zeta)\|_{H^1} + \|\eta\nabla_{\alpha'}\nabla_s^{k-m+1}\zeta\|_{H^1} + \|\eta\nabla_{\alpha'}(\mathbf{J}(\zeta)\nabla_s^{k-m+1}\zeta)\|_{H^1}) \\
&= C_1\Big(\|\zeta\|_{H^k} + \Big\|\eta\nabla_{\alpha'}\Big(\nabla_s^{k-m}\mathcal{D}_\zeta\zeta - \sum_{l=1}^{k-m}\binom{k-m}{l}(\partial_s^l A\nabla_s^{k-m-l+1}\zeta + \partial_s^l C\nabla_s^{k-m-l}\nabla_t\zeta) \\
&\qquad + \sum_{l=1}^{k-m}\binom{k-m}{l}\nabla_s^l(\mathbf{J}(\zeta))\nabla_s^{k-m-l+1}\zeta - \sum_{l=1}^{k-m} C\nabla_s^{l-1}[\nabla_s,\nabla_t]\nabla_s^{k-m-l}\zeta\Big)\Big\|_{H^1} \\
&\qquad + \|\eta\nabla_{\alpha'}\nabla_s^{k-m+1}\zeta\|_{H^1} + \|\eta\nabla_{\alpha'}(\mathbf{J}(\zeta)\nabla_s^{k-m+1}\zeta)\|_{H^1}\Big) \\
&\leq C_1\Big(\|\mathcal{D}_\zeta\zeta\|_{H^k} + \|\zeta\|_{H^k} + \sum_{\substack{\beta\geq 1,\gamma\geq 0,\\ \beta+\gamma=k}}\|\eta\nabla^\beta(\mathbf{J}(\zeta))\nabla^\gamma\nabla_s\zeta\|_{H^0}\Big),
\end{aligned}$$

where in the last inequality we have used the inductive hypothesis to bound $\|\eta\nabla_{\alpha'}\nabla_s^{k-m+1}\zeta\|_{H^1}$.

Step 2a. *In the $k \neq 3$ case, we prove the following inequality:*

$$\sum_{\substack{\beta\geq 1,\gamma\geq 0,\\ \beta+\gamma=k}}\|\eta\nabla^\beta(\mathbf{J}(\zeta))\nabla^\gamma\nabla_s\zeta\|_{H^0} \leq C_1\|\zeta\|_{H^k}. \tag{2.48}$$

It follows from the assumption $k \neq 3$ that if $\beta,\gamma \geq 1$ satisfy $\beta+\gamma = k+1$, then $\min\{\beta,\gamma\} \leq \max\{k-2, 1\}$. Furthermore, the assumption $\|\zeta\|_{\widetilde{H}^k} \leq C_0$ implies the inequality $\|\zeta\|_{C^{k-2}} \leq C_1$ by the embedding of $H^1 \hookrightarrow C^0$ for one-dimensional domains whose lengths are bounded away from zero. This, along with the assumed C^1-bound on ζ, yields (2.48) in the $k \neq 3$ case.

Step 2b. *In the $k = 3$ case, we prove the following inequality:*

$$\sum_{\substack{\beta\geq 1,\gamma\geq 0,\\ \beta+\gamma=3}} \|\eta\nabla_\beta(\mathbf{J}(\zeta))\nabla_\gamma\nabla_s\zeta\|_{H^0} \leq C_1(\|\mathcal{D}_\zeta\zeta\|_{\widetilde{H}^3} + \|\zeta\|_{\widetilde{H}^3} + \delta^{1/2}\|\eta\nabla^3\zeta\|_{H^3}). \tag{2.49}$$

The assumed $\mathcal{C}^1$-bound on ζ implies that the only term in the left-hand side of (2.49) that is not immediately bounded by $C_1\|\zeta\|_{H^3}$ is $\|\eta\nabla^2(\mathbf{J}(\zeta))\nabla\nabla_s\zeta\|_{H^0}$.

Choose smooth maps

$$\begin{aligned} S, U &: \widehat{u}^*\mathrm{T}M_{0211} \to \widehat{u}^*\hom((\mathrm{T}M_{0211})^{\otimes 2}, \mathrm{T}M_{0211}), \\ T &: \widehat{u}^*\mathrm{T}M_{0211} \to \widehat{u}^*\hom((\mathrm{T}M_{0211})^{\otimes 3}, \mathrm{T}M_{0211}), \\ V &: \widehat{u}^*\mathrm{T}M_{0211} \to u^*\hom(\mathrm{T}M_{0211}, \mathrm{T}M_{0211}) \end{aligned}$$

so that the formula

$$\widehat{\nabla}^2(\widehat{J}(\widehat{\zeta})) = S(\widehat{\zeta})(\widehat{\nabla}^2\widehat{\zeta}) + T(\widehat{\zeta})(\widehat{\nabla}\widehat{\zeta}, \widehat{\nabla}\widehat{\zeta}) + U(\widehat{\zeta})(\widehat{\nabla}\zeta) + V(\widehat{\zeta}) \tag{2.50}$$

holds, where the maps S, T, U, V preserve fibers but may not respect their linear structure. Since $\mathbf{J}$ is bounded in $\mathcal{C}^3$, S, T, U, V must be bounded in $\mathcal{C}^1$. We may now use (2.50) to bound the hat-part of $\|\eta\nabla^2(\mathbf{J}(\zeta))\nabla\nabla_s\zeta\|_{H^0}$:

$$\begin{aligned} \|\eta\widehat{\nabla}^2(\widehat{J}(\widehat{\zeta}))\widehat{\nabla}\widehat{\nabla}_s\widehat{\zeta}\|_{H^0} &\leq C_1(\|\widehat{\zeta}\|_{H^2} + \|S(\widehat{\zeta})(\widehat{\nabla}^2\widehat{\zeta})\widehat{\nabla}\widehat{\nabla}_s\widehat{\zeta}\|_{H^0}) \\ &= C_1\big(\|\widehat{\zeta}\|_{H^2} + \|\widehat{\nabla}_s(S(\widehat{\zeta})(\eta\widehat{\nabla}^2\widehat{\zeta})\widehat{\nabla}\widehat{\zeta}) - \widehat{\nabla}_s(S(\widehat{\zeta})(\eta\widehat{\nabla}^2\widehat{\zeta}))\widehat{\nabla}\widehat{\zeta} \qquad (2.51) \\ &\qquad + S(\widehat{\zeta})(\eta\widehat{\nabla}^2)[\widehat{\nabla}, \widehat{\nabla}_s]\widehat{\zeta}\|_{H^0}) \\ &\leq C_1(\|\widehat{\zeta}\|_{H^3} + \delta^{1/2}\|S(\zeta)(\eta\widehat{\nabla}^2\widehat{\zeta})\widehat{\nabla}\widehat{\zeta}\|_{\mathcal{C}^0H^1}) \\ &\leq C_1(\|\widehat{\zeta}\|_{H^3} + \delta^{1/2}\|S(\zeta)(\eta\widehat{\nabla}^2\widehat{\zeta})\|_{\mathcal{C}^0H^1}\|\widehat{\nabla}\widehat{\zeta}\|_{\mathcal{C}^0H^1}), \end{aligned}$$

where in the last inequality we have used the δ-independent Banach algebra property of $\mathcal{C}^0H^1$. By Lemma 2.2.9, $\|\widehat{\nabla}\widehat{\zeta}\|_{\mathcal{C}^0H^1}$ is bounded by $C_1(\|\mathcal{D}_\zeta\zeta\|_{\widetilde{H}^2} + \|\zeta\|_{H^3})$ and therefore by $C_1\|\zeta\|_{\widetilde{H}^3}$; on the other hand, the $\mathcal{C}^1$-bound on S and the $\mathcal{C}^1$-bound on ζ implies the inequality $\|S(\widehat{\zeta})(\eta\widehat{\nabla}^2\widehat{\zeta})\|_{\mathcal{C}^0H^1} \leq C_1\|\eta\widehat{\nabla}^2\widehat{\zeta}\|_{\mathcal{C}^0H^1}$. Substituting these inequalities into (2.51), we obtain:

$$\|\eta\widehat{\nabla}^2(\widehat{J}(\widehat{\zeta}))\widehat{\nabla}\widehat{\nabla}_s\widehat{\zeta}\|_{H^0} \leq C_1(\|\zeta\|_{H^3} + \delta^{1/2}\|\zeta\|_{\widetilde{H}^3}\|\eta\nabla^2\zeta\|_{\mathcal{C}^0H^1}) \leq C_1(\|\zeta\|_{H^3} + \delta^{1/2}\|\eta\nabla^2\zeta\|_{\mathcal{C}^0H^1}). \tag{2.52}$$

Next, we use Lemma 2.2.9 to bound $\|\eta\nabla^2\zeta\|_{\mathcal{C}^0H^1}$:

$$\begin{aligned} \|\eta\nabla^2\zeta\|_{\mathcal{C}^0H^1} &\leq C_1(\|\zeta\|_{\widetilde{H}^3} + \|\nabla^2(\eta\zeta)\|_{\mathcal{C}^0H^1}) \\ &\leq C_1(\|\eta\zeta\|_{H^4} + \|\nabla\mathcal{D}_\zeta(\eta\zeta)\|_{\mathcal{C}^0H^1} + \|\zeta\|_{\widetilde{H}^3}) + P(\|\nabla\zeta\|_{\mathcal{C}^0H^1})(\|\zeta\|_{H^3_{\delta,\rho}} + \|\mathcal{D}_\zeta\zeta\|_{\mathcal{C}^0H^1}) \qquad (2.53) \\ &\leq C_1(\|\mathcal{D}_\zeta\zeta\|_{\widetilde{H}^3} + \|\zeta\|_{\widetilde{H}^3} + \|\eta\nabla^3\zeta\|_{H^1}) + P(\|\zeta\|_{\widetilde{H}^3})\|\zeta\|_{\widetilde{H}^3} \\ &\leq C_1(\|\mathcal{D}_\zeta\zeta\|_{\widetilde{H}^3} + \|\zeta\|_{\widetilde{H}^3} + \|\eta\nabla^3\zeta\|_{H^1}), \end{aligned}$$

where the last inequality follows from the assumed bound on $\|\zeta\|_{\widetilde{H}^3}$. Substituting (2.53) into (2.52), we obtain:

$$\|\eta\widehat{\nabla}^2(\widehat{J}(\widehat{\zeta}))\widehat{\nabla}\widehat{\nabla}_s\widehat{\zeta}\|_{H^0} \leq C_1(\|\zeta\|_{H^3} + \delta^{1/2}(\|\mathcal{D}_\zeta\zeta\|_{\widetilde{H}^3} + \|\zeta\|_{\widetilde{H}^3} + \|\eta\nabla^3\zeta\|_{H^1})) \leq C_1(\|\mathcal{D}_\zeta\zeta\|_{\widetilde{H}^3} + \|\zeta\|_{\widetilde{H}^3} + \delta^{1/2}\|\eta\nabla^3\zeta\|_{H^1}). \tag{2.54}$$

To bound the 02-part of $\|\eta\nabla^2(\mathbf{J}(\zeta))\nabla\nabla_s\zeta\|_{H^0}$, we use the the fact that the domains $Q_{02,\delta,r}$ satisfy a uniform cone condition:

$$\|\eta\nabla^2_{02}(J_{02}(\zeta_{02}))\nabla_{02}\nabla_{02,s}\zeta_{02}\|_{H^0)} \overset{\text{Hölder}}{\leq} C_1\|\nabla^2_{02}(J_{02}(\zeta_{02}))\|_{L^4}\|\nabla^2_{02}\zeta\|_{L^4} \leq C_1(1+\|\zeta\|_{H^3})\|\zeta\|_{H^3}, \tag{2.55}$$

where the second inequality follows from the Sobolev embedding $H^1 \hookrightarrow L^4$ for two-dimensional domains satisfying a cone condition. Combining (2.54) and (2.55) and using the assumed bound on $\|\zeta\|_{\widetilde{H}^3}$ yields the desired bound:

$$\|\eta\nabla^2(\mathbf{J}(\zeta))\nabla\nabla_s\zeta\|_{H^0} \leq C_1(\|\mathcal{D}_\zeta\zeta\|_{\widetilde{H}^3} + \|\zeta\|_{\widetilde{H}^3} + \delta^{1/2}\|\eta\nabla^3\zeta\|_{H^1}).$$

Step 3. *We prove Lemma 2.2.11.*

The $k \neq 3$ case of Lemma 2.2.11 is an immediate consequence of Steps 1b and 2a.

Toward the $k = 3$ case of Lemma 2.2.11, let us show that there exists $\delta_0 \in (0, r_1]$ such that (2.46) holds for $\delta \in (0, \delta_0]$. Combining (2.47) and (2.49) yields the following inequality:

$$\|\eta\nabla^3\zeta\|_{H^1} \leq C_1(\|\mathcal{D}_\zeta\zeta\|_{\widetilde{H}^3} + \|\zeta\|_{\widetilde{H}^3} + \delta^{1/2}\|\eta\nabla^3\zeta\|_{H^1}). \tag{2.56}$$

If we set $\delta_0 := \min\{(2C_1)^{-2}, r_1\}$, where C_1 is the constant appearing in (2.56), then (2.56) yields the uniform inequality $\|\eta\nabla^3\zeta\|_{H^1} \leq C_1(\|\mathcal{D}_\zeta\zeta\|_{\widetilde{H}^3} + \|\zeta\|_{\widetilde{H}^3})$ for all $\delta \in (0, \delta_0]$.

It remains to establish the $k = 3$ case of (2.46) for $\delta \in [\delta_0, r_1]$. To do so, we begin by bounding $\|\nabla^2(\mathbf{J}(\zeta))\nabla^2\zeta\|_{H^0}$, using the fact that the domains $\mathbf{Q}_{\delta,r}$ satisfy a uniform cone condition for $\delta \in [\delta_0, r_1/4]$:

$$\|\nabla^2(\mathbf{J}(\zeta))\nabla^2\zeta\|_{H^0} \overset{\text{Hölder}}{\leq} C_1\|\nabla^2(\mathbf{J}(\zeta))\|_{L^4}\|\nabla^2\zeta\|_{L^4} \overset{\text{Sobolev}}{\leq} C_1(1+\|\zeta\|_{H^{2,4}})\|\zeta\|_{H^{2,4}} \leq C_1(1+\|\zeta\|_{H^3})\|\zeta\|_{H^3} \leq C_1\|\zeta\|_{H^3}. \tag{2.57}$$

Substituting (2.57) into (2.47) yields the $k = 3$ case of (2.46) for $\delta \in [\delta_0, r_1/4]$:

$$\|\eta\nabla^3\zeta\|_{H^1} \leq C_1\Big(\|\mathcal{D}_\zeta\zeta\|_{H^3} + \|\zeta\|_{H^3} + \sum_{\substack{\beta\geq 1,\gamma\geq 0,\\ \beta+\gamma=3}} \|\eta\nabla^\beta(\mathbf{J}(\zeta))\nabla^\gamma\nabla_s\zeta\|_{H^0}\Big) \leq C_1(\|\mathcal{D}_\zeta\zeta\|_{H^3} + \|\zeta\|_{H^3}).$$

□

Proof of lemma 2.2.8. Lemma 2.2.8 follows immediately from Lemmata 2.2.9 and 2.2.11. Indeed, choose $\eta\colon \overline{Q}_{02,\delta,r_2} \to \mathbb{R}$ to be a smooth function with $\eta|_{\overline{Q}_{02,\delta,r_1}} \equiv 1$ and $\operatorname{supp}\eta \subset Q_{02,\delta,r_2}$. C_1 and P will denote a δ-independent constant and a δ-independent polynomial that may change from line to line. Lemma 2.2.11 yields a bound on $\|\zeta\|_{H^{k+1}(\mathbf{Q}_{\delta,r_1})}$:

$$\|\zeta\|_{H^{k+1}(\mathbf{Q}_{\delta,r_1})} \leq \|\eta\zeta\|_{H^{k+1}(\mathbf{Q}_{\delta,r_2})} \leq C_1\big(\|\zeta\|_{\widetilde{H}^k(\mathbf{Q}_{\delta,r_2})} + \|\mathcal{D}_\zeta\zeta\|_{\widetilde{H}^k(\mathbf{Q}_{\delta,r_2})}\big). \tag{2.58}$$

Lemma 2.2.9 yields a bound on $\sum_{l=0}^{k-1}\|\nabla^l\zeta\|_{C^0H^1(\mathbf{Q}_{\delta,r_1})}$:

$$\sum_{l=0}^{k-1}\|\nabla^l\zeta\|_{C^0H^1(\mathbf{Q}_{\delta,r_1})} \leq C_1\big(\|\zeta\|_{H^{k+1}(\mathbf{Q}_{\delta,r_1})}+\|\mathcal{D}_\zeta\zeta\|_{\widetilde{H}^k(\mathbf{Q}_{\delta,r_1})}\big)+ \tag{2.59}$$

$$+P\big(\|\zeta\|_{\widetilde{H}^k(\mathbf{Q}_{\delta,r_1})}\big)\cdot\big(\|\zeta\|_{H^k(\mathbf{Q}_{\delta,r_1})}+\|\mathcal{D}_\zeta\zeta\|_{\widetilde{H}^{k-1}(\mathbf{Q}_{\delta,r_1})}\big) \tag{2.60}$$

$$\overset{(2.58)}{\leq} C_1\big(\|\mathcal{D}_\zeta\zeta\|_{\widetilde{H}^k(\mathbf{Q}_{\delta,r_2})}+\|\zeta\|_{\widetilde{H}^k(\mathbf{Q}_{\delta,r_2})}\big),$$

where in the second inequality we have used the assumed bound on $\|\zeta\|_{\widetilde{H}^k(\mathbf{Q}_{\delta,r_1})}$. Combining (2.58) and (2.59) yields $\|\zeta\|_{\widetilde{H}^{k+1}(\mathbf{Q}_{\delta,r_1})}\leq C_1(\|\mathcal{D}_\zeta\zeta\|_{\widetilde{H}^k(\mathbf{Q}_{\delta,r_2})}+\|\zeta\|_{\widetilde{H}^k(\mathbf{Q}_{\delta,r_2})})$, which can be used to inductively prove the desired inequality (2.38). □

We will not use the following proposition in this paper. However, it will be used in [BoWe1] to show that the linearized Cauchy–Riemann operator defines a Fredholm section.

Proposition 2.2.12 (linear elliptic estimate for $k=2$). *There is a constant $\epsilon>0$ and for every $C_0>0$, $k\geq 0$, and $0<r_1<r_2<\rho$ there is a constant C_1 such that the inequality*

$$\|\zeta\|_{H^{k+1}(\mathbf{Q}_{\delta,r_1})}\leq C_1\big(\|\mathcal{D}_\xi\zeta\|_{H^k(\mathbf{Q}_{\delta,r_2})}+\|\zeta\|_{H^0(\mathbf{Q}_{\delta,r_2})}\big)$$

holds for any choice of $\delta\in(0,r_1/4]$, a coherent collection $\mathbf{j}$ of complex structures on $\overline{\mathbf{Q}}_{\delta,\rho}$ with $\|\mathbf{j}-\mathbf{i}\|_{C^0}\leq\epsilon$ and $\|\mathbf{j}-\mathbf{i}\|_{C^2}\leq C_0$, a pair $\mathbf{J}$ of compatible almost complex structures on $\overline{\mathbf{Q}}_{\delta,\rho}$ which are contained in a C^2-ball of radius C_0 and which induce by (2.3) whose pairwise constants of equivalence are bounded above by C_0, and two pairs of sections $\zeta,\xi\in\Gamma^{k+2}_{u_\delta}(\mathbf{Q}_{\delta,r_2})$ with $\|\xi\|_{C^0}\leq\epsilon$ and $\|\xi\|_{C^1}\leq C_0$.

The proof is an easier version of the proof of Lemma 2.2.8.

Chapter 3

Compactness

In this chapter, which is joint with Katrin Wehrheim, we establish a Gromov compactness theorem for strip shrinking in pseudoholomorphic quilts when composition of Lagrangian correspondences is immersed. In particular, we show that figure eight bubbling occurs in the limit, argue that this is a codimension-0 effect, and predict its algebraic consequences — geometric composition extends to a curved A_∞-bifunctor, in particular the associated Floer complexes are isomorphic after a figure eight correction of the bounding cochain.

Philosophical Remark: *In the early days of pseudoholomorphic quilts, the relevance of figure eight bubbles was doubted. We hope that this chapter puts those doubts to rest. As we summarize in §3.1.2 and explain in §3.4, figure eight bubbling cannot be* a priori *excluded for dimension reasons, and will contribute to the algebra. So even if e.g. the isomorphism* (2.1) *of Floer homology under geometric composition was to be proven with methods other than strip shrinking, one must in general expect figure eight type obstructions. However, figure eight bubbles should be viewed as a tool rather than an inconvenience: For instance, they provide a cochain for $L_{01} \circ L_{12}$ so that* (2.1) *continues to hold in more general situations than [WeWo1] considered. More generally, fully embracing figure eight bubbles will yield a natural 2-categorical structure on the collection of all compact symplectic manifolds, which will unify and extend a wide variety of currently-known algebraic structures.*

While this chapter only provides substantial evidence for these algebraic results (or "proofs up to technical details" depending on ones standards of rigour), it does demonstrate in full detail that figure eight bubbling is in fact analytically manageable: Section 3.2 gives rigorous definitions of "squiggly strip shrinking" and the novel bubble types — figure eight bubbles and squashed eight bubbles — and establishes lower bounds as well as topological controls on their energy. A full removal of singularities for the new bubbles is established in [Bo1]. Section 3.3 shows how the full diversity of bubble types appears in the Gromov compactification for "squiggly strip shrinking". Moreover, Appendix B in collaboration with Felix Schmäschke provides the first nontrivial example of a general figure eight bubble.

3.1 Introduction

3.1.1 Analytic Results

The compactness analysis in §3.3 will for ease of notation be performed in the special case of quilted squares with seam conditions in L_{01}, L_{12} and the width of the strip mapping to M_1

converging to zero. However, it generalizes directly to the following result for strip or annulus shrinking in pseudoholomorphic quilts. (For an introduction to quilts, see [WeWo3].)

Gromov Compactness Theorem 3.3.1: Let $\underline{Q}^\nu$ be a sequence of quilted surfaces containing a patch Q_1^ν diffeomorphic to an annulus or strip, equipped with complex structures in which the width of Q_1^ν tends to zero as $\nu \to \infty$. (For allowable squiggliness — i.e. variation of width — see Definition 3.2.2.) Label the patches of $\underline{Q}^\nu$ with a tuple $\underline{M}$ of closed symplectic manifolds (or noncompact ones without boundary which come with a priori $\mathcal{C}^0$-bounds as discussed in Remark 3.3.4), let M_1 and M_0, M_2 be the labels of Q_1^ν and the adjacent patches, and fix compatible almost complex structures over each patch. Fix compact Lagrangian seam conditions for each seam of $\underline{Q}^\nu$ so that the Lagrangian correspondences L_{01}, L_{12} associated to the seams bordering Q_1^ν have immersed composition $L_{01} \circ L_{12}$. Now suppose that $(\underline{v}^\nu)_{\nu\in\mathbb{N}} : \underline{Q}^\nu \to \underline{M}$ is a sequence of pseudoholomorphic quilts of bounded energy with the given seam conditions.

Then there is a subsequence (still denoted $(\underline{v}^\nu)_{\nu\in\mathbb{N}}$) that converges up to bubbling to a punctured pseudoholomorphic quilt $\underline{v}^\infty : \underline{Q}^\infty \smallsetminus Z \to (\underline{M} \smallsetminus M_1)$. Here $\underline{Q}^\infty$ is the quilted surface obtained as limit of the $\underline{Q}^\nu$ by replacing Q_1^ν with a seam labeled by $L_{01} \circ L_{12}$, Z is a finite set of bubbling points, $\underline{v}^\infty$ satisfies seam conditions in the fixed Lagrangian correspondences and for the new seam in $L_{01} \circ L_{12}$ (in the generalized sense of (3.2)), and convergence holds in the following sense:

- The energy densities $|\mathrm{d}\underline{v}^\nu|^2$ are uniformly bounded on every compact subset of $\underline{Q}^\infty \smallsetminus Z$, and at each point in Z there is energy concentration of at least $\hbar > 0$, given by the minimal bubbling energy from Definition 3.2.7.
- The quilt maps $\underline{v}^\nu|_{\underline{Q}^\nu \smallsetminus (Q_1^\nu \cup Z)}$ on the complement of Z in the patches other than Q_1^ν converge with all derivatives on every compact set to $\underline{v}^\infty$.
- At least one type of bubble forms at each point $z \in Z$ in the following sense: There is a sequence of (tuples of) maps obtained by rescaling the maps defined on the various patches near z, which converges $\mathcal{C}^\infty_{\mathrm{loc}}$ to one of the following:
  - a nonconstant, finite-energy pseudoholomorphic map $\mathbb{R}^2 \to M_\ell$ to one of the symplectic manifolds in $\underline{M}$ (this can be completed to a nonconstant pseudoholomorphic sphere in M_ℓ);
  - a nonconstant, finite-energy pseudoholomorphic map $\mathbb{H} \to M_k^- \times M_\ell$ to a product of symplectic manifolds associated to the patches on either side of a seam in $\underline{Q}^\nu$, that satisfies the corresponding Lagrangian seam condition (this can be extended to a nonconstant pseudoholomorphic disk in $M_k^- \times M_\ell$, in particular including the cases of disks with boundary on $L_{01} \subset M_0^- \times M_1$ or $L_{12} \subset M_1^- \times M_2$);
  - a nonconstant, finite-energy figure eight bubble in the sense of (3.1) below;
  - a nonconstant, finite-energy squashed eight bubble in the sense of (3.2) below, with generalized seam conditions in $L_{01} \circ L_{12}$.

Gromov compactification of strip-shrinking moduli spaces: Note that the above partial Gromov compactness statement only requires the composition $L_{01} \circ L_{12}$ to be immersed. If the self-intersections of this immersion are locally clean, then the results of [Bo1] allow us to remove the singularities in the limits of the main component as well as the figure eight and squashed eight bubbles. Moreover, the techniques of [Bo1] also provide "bubbles connect" results for long cylinders, so that a "soft rescaling iteration" (guided by capturing all energy) will yield the following full Gromov compactification of a moduli space with squiggly strip- (or annulus-) shrinking in terms of bubble trees: On the complement of the new seam, these trees are made up of trees of disk bubbles[1] attached to the seams, with additional trees of sphere bubbles attached to the disks, seams, or interior of the patches. On the new seam, as indicated in Figure 1-2, starting from the root, every bubble tree starts with a (possibly empty or containing constant vertices) tree of squashed eight bubbles. Attached to this are figure eight bubbles (possibly constant) in such a way that between any leaf and the root of the complete tree there is at most one figure eight. Trees of disk bubbles with boundary on L_{01} resp. L_{12} can then be attached to the corresponding seams of the figure eight bubbles. Finally, trees of sphere bubbles can be attached to the interior of each patch or the seams in this bubble tree. The hierarchy in this compactification is illustrated in Figure 3-2 (including the additional complication of Morse flow lines). We will not provide a detailed construction of this compactification in the present chapter, whose point is to establish the foundational analysis. The full compactification (including Morse flow lines) will be part of the polyfold setup in[BoWe1].

Singular quilt bubbling phenomena: Beyond the standard bubbling phenomena (holomorphic spheres and disks) our Gromov compactification for quilts with strip-shrinking involves two new types of bubbles: A **figure eight bubble** is a tuple of finite-energy pseudoholomorphic maps

$$w_0\colon \mathbb{R}\times(-\infty,-\tfrac{1}{2}]\to M_0,\qquad w_1\colon \mathbb{R}\times[-\tfrac{1}{2},\tfrac{1}{2}]\to M_1,\qquad w_2\colon \mathbb{R}\times[\tfrac{1}{2},\infty)\to M_2 \tag{3.1}$$

satisfying the seam conditions

$$(w_0(s,-\tfrac{1}{2}),w_1(s,-\tfrac{1}{2}))\in L_{01},\quad (w_1(s,\tfrac{1}{2}),w_2(s,\tfrac{1}{2}))\in L_{12}\qquad \forall\, s\in\mathbb{R},$$

while a **squashed eight bubble** is a triple of finite-energy pseudoholomorphic maps

$$w_0\colon \mathbb{R}\times(-\infty,0]\to M_0,\qquad w_1\colon \mathbb{R}\to M_1,\qquad w_2\colon \mathbb{R}\times[0,\infty)\to M_2$$

satisfying the seam condition

$$(w_0(s,0),w_1(s),w_1(s),w_2(s,0))\in L_{01}\times_{M_1}L_{12}\quad \forall\, s\in\mathbb{R}. \tag{3.2}$$

Remark 3.1.1. Both of these bubbles are of singular quilt type in the sense that we cannot generally expect a smooth extension to a quilted sphere. For the squashed eight bubble this results from the boundary condition $L_{01}\circ L_{12}$ generally just being a Lagrangian immersion. For the figure eight bubble this is due to the way in which the two seams intersect at infinity: After stereographic compactification to a quilted sphere, they touch tangentially rather than intersect transversely, which would allow a description of the singularity in terms of striplike ends. Nevertheless, if the composition $L_{01}\circ L_{12}$ is cleanly immersed, then the removable singularity result in [Bo1] shows that $w_0(s,t)\to p_0\in M_0$ and $w_2(s,t)\to p_2\in M_2$ have uniform limits as $s^2+t^2\to\infty$, whereas $w_1(s,t)\to p_1^{\pm}\in M_1$ has two possibly different limits as $s\to\pm\infty$, both of which are lifts of $(p_0,p_2)\in L_{01}\circ L_{12}$, that is $(p_0,p_1^{\pm},p_1^{\pm},p_2)\in L_{01}\times_{M_1}L_{12}$.

[1] Here we identify spheres with a single circle as seam and two patches labeled by M_k and M_ℓ with disk bubbles in $M_k^-\times M_\ell$ by "folding" across the seam as in the portion of the proof of Theorem 3.3.1 treating the (D01) case.

Recall that even smooth removal of singularity generally does not provide lower bounds on the energy of bubbles except in situations with simple topology; see Remark 3.2.10. In §3.2 we establish this lower energy bound for figure eight and squashed eight bubbles by purely analytic means.

Lower Energy Bound Lemma 3.2.8: For fixed almost complex structures and Lagrangians with immersed composition $L_{01} \circ L_{12}$, the energy of nontrivial figure eight and squashed eight bubbles is bounded below by a positive quantity.

Finally, Appendix B in collaboration with Felix Schmäschke explains how pseudoholomorphic disks and strips can be viewed as special cases of figure eight bubbles, and we provide an example of a nontrivial figure eight bubble with embedded composition $L_{01} \circ L_{12}$ and target spaces $M_0 = M_2 = \mathbb{CP}^3$, $M_1 = \mathbb{CP}^1 \times \mathbb{CP}^1$.

3.1.2 Algebraic consequences of figure eight bubbling

We establish our analytic results in settings that will allow us to describe compactified moduli spaces of pseudoholomorphic quilts with shrinking strips as zero sets of Fredholm sections in polyfold bundles. This will put the universal regularization theory of [HoWyZe2] at our disposal. In particular, there will be no need to prove a separate gluing theorem for exhibiting configurations with figure eight bubbles as boundaries of the compactified moduli space: Pre-gluing constructions outlined in §3.4.1 will provide polyfold charts with boundary for bubble trees of (not necessarily pseudoholomorphic) quilted maps, and the proof of the nonlinear Fredholm property of the quilted Cauchy–Riemann operator in this polyfold setup will be essentially a version of the quadratic estimates in the classical gluing analysis (which in our case should follow from combining the results of [WeWo1] and [Bo1]). Moreover, the boundary stratification of the ambient polyfold will directly induce the boundary stratification of the (regularized) moduli spaces. This offers a **semi-rigorous method for predicting algebraic consequences:** If the considered moduli spaces can be cut out from ambient polyfolds, then the algebraic identities are given by summing over the top boundary strata of the polyfold. Furthermore, if the local charts for the polyfold arise from pre-gluing constructions (as has been the case in all known examples), then the boundary stratification can be read off from the gluing parameters. We thus analyze in §3.4.2 the boundary strata predicted by our Gromov-compactification in the case of strip-shrinking used to prove (2.1). Based on that, §3.4.3 gives a fair amount of detail on the extension of these moduli spaces by Morse trajectories. This is desirable for easy polyfold implementation as well as reducing algebraic headaches by working with finitely generated chain complexes. Finally, we use this analysis of boundary strata to predict in §3.4.4 a generalization of the isomorphism (2.1) of Floer homology under geometric composition.

Besides a Fredholm description of figure eight moduli spaces and generalization of (2.1), another motivation for developing the analysis described in §3.1.1 is to obtain a new approach to the construction of the A_∞-functors associated to monotone Lagrangian correspondences in [MaWeWo]. Whereas the latter requires a technically cumbersome construction of quilted surfaces with striplike ends and regular Hamiltonian perturbations, and is heavily restricted by monotonicity requirements, a polyfold setup for figure eight moduli spaces will provide a direct construction of curved[2] A_∞-functors for general Lagrangian correspondences. As

[2] Disk bubbling gives rise to m_0 terms in all A_∞-relations, so that in particular squares of differentials can be nonzero. This can be thought of as allowing curvature, hence we follow e.g. [Au] and denote this generalized type of A_∞-relations by the prefix "curved".

we have explained in Chapter 1, we will be able to work directly with the singular quilted surfaces that realize the multiplihedra in [MaWo], since these are special cases of figure eights with $M_0 = \mathrm{pt}$ and marked points on boundary and seams. Analyzing moreover the boundary stratification of general figure eight moduli spaces, we arrive at the conjecture that geometric composition is encoded in terms of a curved A_∞-bifunctor, which in turn specializes to the desired generalization of the A_∞-functors in [MaWeWo]. In fact, these methods can be extended to generalizations of figure eight moduli spaces, leading to a conjectural symplectic $(\infty, 2)$-category that we will further investigate in future work. While the complete polyfold construction of these new algebraic structures in [BoWel] will be lengthy since we aim to provide a technically sound and easily portable basis for all future use of quilt moduli spaces, its rough form and algebraic consequences are already so apparent from our current understanding that it seems timely to give this semi-rigorous exposition. We do so in order to motivate the development of this theory and enable investigations of its future applications.

3.2 Squiggly strip quilts and figure eight bubbles

The purpose of this section is to establish a general setup for squiggly strip shrinking in quilted surfaces and introduce the new bubbling phenomena with their basic properties. Besides sphere and disk bubbling, two novel sorts of bubbles may appear: figure eight bubbles and squashed eight bubbles, both of which are introduced in Definition 3.2.5. In Lemma 3.2.8, we show that the energy of the figure eight and squashed eight bubbles is bounded below, which will be a key ingredient in our proof of the Gromov Compactness Theorem 3.3.1. The proof of Lemma 3.2.8 relies on a $\mathcal{C}^\infty$-compactness statement for squiggly strip shrinking from [Bo1], which we restate in Theorem 3.2.9.

In this section and the next we will be working with symplectic manifolds and with pseudoholomorphic curves with seam conditions defined by compact Lagrangian correspondences

$$L_{01} \subset M_0^- \times M_1, \qquad L_{12} \subset M_1^- \times M_2. \tag{3.3}$$

When the following intersection in $M_0^- \times M_1 \times M_1^- \times M_2$ is transverse, we follow [WeWo4] and say that L_{01} and L_{12} have **immersed composition**:

$$(L_{01} \times L_{12}) \pitchfork (M_0^- \times \Delta_1 \times M_2) \; =: \; L_{01} \times_{M_1} L_{12}. \tag{3.4}$$

Indeed, the transversality implies that $L_{01} \times_{M_1} L_{12} \subset M_0^- \times M_1 \times M_1^- \times M_2$ is a compact submanifold, and the projection $\pi_{02} \colon L_{01} \times_{M_1} L_{12} \to M_0^- \times M_2$ is a Lagrangian immersion by e.g. [WeWo4, Lemma 2.0.5] (which builds on [GuSt, §4.1]). We will denote its image, the **geometric composition** of L_{01} and L_{12}, by

$$L_{01} \circ L_{12} \; := \; \pi_{02}(L_{01} \times_{M_1} L_{12}) \; \subset \; M_0^- \times M_2.$$

In some contexts we will assume **cleanly-immersed composition**, that is an immersed composition such that any two local branches of the immersed Lagrangian $L_{01} \circ L_{12}$ intersect cleanly.

Throughout §3.2 and §3.3 we will work with fixed symplectic manifolds M_0, M_1, M_2 without boundary which are either compact or satisfy boundedness assumptions as detailed in Remark 3.3.4, and compact Lagrangians L_{01}, L_{12} as in (3.3) with immersed composition.

We will consider pseudoholomorphic quilts with respect to compatible almost complex structure:

$$J_\ell \colon [-\rho,\rho]^2 \to \mathcal{J}(M_\ell,\omega_\ell) \qquad \text{for } \ell = 0,1,2. \tag{3.5}$$

These are allowed to be domain-dependent[3] but are $\mathcal{C}^k$ as maps $[-\rho,\rho]^2 \times \mathrm{T}M_\ell \to \mathrm{T}M_\ell$, where k will be either a positive integer or infinity. Then compatibility means that

$$g_\ell(s,t) := \omega_\ell(-, J_\ell(s,t)-) \tag{3.6}$$

are metrics on M_ℓ that are $\mathcal{C}^k$ in $(s,t) \in [-\rho,\rho]^2$.

The underlying quilted domains of our pseudoholomorphic quilt maps will be open squares with two seams. One should imagine these as part of the domain of a larger pseudoholomorphic quilt with compact domain or with quilted cylindrical ends. The basic (localized and rescaled) examples studied in [WeWo1] are squares $(-1,1)^2$ with seams $(-1,1)\times\{\pm\delta\}$, whose main feature is a middle strip $(-1,1)\times[-\delta,\delta]$ of constant width $2\delta > 0$. The following definition generalizes the underlying quilted surfaces to allow middle domains $\{|t| \le f(s)\}$ of local widths $2f(s) > 0$ varying with $s \in (-1,1)$. Since diffeomorphically such domains are still strips, we call them "squiggly strips".

Definition 3.2.1. Fix $\rho > 0$, a real-analytic function $f \colon [-\rho,\rho] \to (0,\rho/2]$, almost complex structures J_ℓ, $\ell = 0,1,2$ as in (3.5), and a complex structure j on $[-\rho,\rho]^2$. A $(\mathbf{J_0},\mathbf{J_1},\mathbf{J_2},\mathbf{j})$**-holomorphic size-**$(\mathbf{f},\rho)$ **squiggly strip quilt for** $(\mathbf{L_{01}},\mathbf{L_{12}})$ is a triple of smooth maps

$$\underline{v} = \begin{pmatrix} v_0 \colon \{(s,t) \in (-\rho,\rho)^2 \mid t \le -f(s)\} \to M_0 \\ v_1 \colon \{(s,t) \in (-\rho,\rho)^2 \mid |t| \le f(s)\} \to M_1 \\ v_2 \colon \{(s,t) \in (-\rho,\rho)^2 \mid t \ge f(s)\} \to M_2 \end{pmatrix} \tag{3.7}$$

that satisfy the Cauchy–Riemann equations

$$\mathrm{d}v_\ell(s,t) \circ j(s,t) - J_\ell(s,t,v_\ell(s,t)) \circ \mathrm{d}v_\ell(s,t) = 0 \qquad \forall\, \ell = 0,1,2 \tag{3.8}$$

for (s,t) in the relevant domains, fulfill the seam conditions

$$\big(v_0(s,-f(s)), v_1(s,-f(s))\big) \in L_{01}, \qquad \big(v_1(s,f(s)), v_2(s,f(s))\big) \in L_{12} \qquad \forall\, s \in (-\rho,\rho), \tag{3.9}$$

and have finite energy[4]

$$E(\underline{v}) \;:=\; \textstyle\int v_0^*\omega_0 + \int v_1^*\omega_1 + \int v_2^*\omega_2 \;=\; \frac{1}{2}\Big(\int |\mathrm{d}v_0|^2 + \int |\mathrm{d}v_1|^2 + \int |\mathrm{d}v_2|^2\Big) \;<\; \infty.$$

When j is the standard complex structure i, (3.8) reduces to the equation

$$\partial_s v_\ell(s,t) + J_\ell(s,t,v_\ell(s,t))\partial_t v_\ell(s,t) = 0.$$

When considering a (J_0,J_1,J_2)-holomorphic squiggly strip quilt $\underline{v}^\nu$, it will be useful to

[3] Note in particular that we do not require J_ℓ to be constant near the seam as in [WeWo1]. This is necessary because the corrected proof of transversality in [WeWo5] does not guarantee regular almost complex structures in this class.

[4] Throughout we will make use of the standard energy identity [McSa, Lemma 2.2.1] for pseudoholomorphic maps.

consider the **energy density functions**

$$|\mathrm{d}\underline{v}| : (-1,1)^2 \to [0,\infty), \qquad |\mathrm{d}\underline{v}(s,t)| := \left(\left|\mathrm{d}v_0(s,t)\right|_{J_0}^2 + \left|\mathrm{d}v_1(s,t)\right|_{J_1}^2 + \left|\mathrm{d}v_2(s,t)\right|_{J_2}^2 \right)^{\frac{1}{2}}, \tag{3.10}$$

where the norms $|\mathrm{d}v_\ell(s,t)|$ are induced by the metrics defined by ω_ℓ and J_ℓ as in (3.6) and set to be zero on the complement of the domain of v_ℓ. If $\underline{v}$ is a squiggly strip quilt, then $|\mathrm{d}\underline{v}|$ is upper semi-continuous on $\{(s, \pm f(s))\}$, continuous elsewhere, and satisfies $E(\underline{v}) = \frac{1}{2}\int |\mathrm{d}\underline{v}|^2$.

The goal of this chapter is to generalize and strengthen the strip shrinking analysis in [WeWo1], which considers sequences of $(J_0^\nu, J_1^\nu, J_2^\nu)$-holomorphic squiggly strip quilts of width $f^\nu \equiv \delta^\nu \to 0$. For that purpose we consider varying width functions f^ν that uniformly converge to zero as follows.

Definition 3.2.2. Fix $\rho > 0$. A sequence $\left(f^\nu\right)_{\nu\in\mathbb{N}}$ of real-analytic functions $f^\nu\colon [-\rho,\rho] \to (0,\rho/2]$ **obediently shrinks to zero, $\mathbf{f^\nu \Rightarrow 0}$,** if $\max_{s\in[-\rho,\rho]} f^\nu(s) \underset{\nu\to\infty}{\longrightarrow} 0$ and

$$\sup_{\nu\in\mathbb{N}} \frac{\max_{s\in[-\rho,\rho]} \left|\frac{\mathrm{d}^k}{\mathrm{d}s^k} f^\nu(s)\right|}{\min_{s\in[-\rho,\rho]} f^\nu(s)} =: C_k < \infty \qquad \forall\, k \in \mathbb{N}_0,$$

and in addition there are holomorphic extensions $F^\nu\colon [-\rho,\rho]^2 \to \mathbb{C}$ of $f^\nu(s) = F^\nu(s,0)$ such that (F^ν) converges $\mathcal{C}^\infty$ to zero[5].

We will see in Theorem 3.3.1 that any sequence $(\underline{v}^\nu)$ of pseudoholomorphic squiggly strip quilts of bounded energy and obediently shrinking widths $f^\nu \Rightarrow 0$ has a subsequence that — up to finitely many points where energy concentrates — converges to a degenerate strip quilt, in which the middle domain mapping to M_1 is replaced by a single straight seam mapping to the immersed Lagrangian $L_{01} \circ L_{12}$. Here bubbling near the middle squiggly strip may lead to limit maps whose seam values switch between the sheets of $L_{01} \circ L_{12}$. Thus we need to allow for singularities in the degenerate strip quilts as follows.

Definition 3.2.3. Fix $\rho > 0$, almost complex structures J_ℓ, $\ell \in \{0,2\}$ as in (3.5), and a complex structure j on $[-\rho,\rho]^2$. A **$(\mathbf{J_0}, \mathbf{J_2}, \mathbf{j})$-holomorphic size-$\rho$ degenerate strip quilt for $(\mathbf{L_{01}} \times_{\mathbf{M_1}} \mathbf{L_{12}})$ with singularities** is a triple of smooth maps

$$\underline{v} = \begin{pmatrix} u_0\colon\ (-\rho,\rho) \times (-\rho,0] \smallsetminus S \times \{0\} \to M_0 \\ u_1\colon\ (-\rho,\rho) \smallsetminus S \to M_1 \\ u_2\colon\ (-\rho,\rho) \times [0,\rho) \smallsetminus S \times \{0\} \to M_2 \end{pmatrix}$$

defined on the complement of a finite set $S \subset \mathbb{R}$ that satisfy the Cauchy–Riemann equation (3.8) for $\ell \in \{0,2\}$ and (s,t) in the relevant domains, fulfill the lifted seam condition

$$\big(u_0(s,0), u_1(s), v_1(s), u_2(s,0)\big) \in L_{01} \times_{M_1} L_{12} \qquad \forall\, s \in (-\rho,\rho) \smallsetminus S,$$

[5]To see why the last condition is necessary, consider the sequence of functions $(F^\nu\colon [-1,1]^2 \to \mathbb{C})$ defined by $F^\nu(z) := \exp(-\nu)\sin(2\nu z) + 1/4$. For $x \in \mathbb{R}$, we have the formulas $F^\nu(x) = \exp(-\nu)\sin(2\nu x) + 1/4$ and $F^\nu(ix) = i\exp(-\nu)(\exp(2\nu x) - \exp(-2\nu x))/2 + 1/4$, so the restrictions to $[-1,1] \times \{0\}$ converge in $\mathcal{C}^\infty$ to zero but $F^\nu(3i/4)$ diverges to $i\infty$.

and have finite energy

$$E(\underline{u}) := \int u_0^*\omega_0 + \int u_2^*\omega_2 = \tfrac{1}{2}\Big(\int|\mathrm{d}u_0|^2 + \int|\mathrm{d}u_2|^2\Big) < \infty.$$

Remark 3.2.4. If u_1 in the above definition continuously extends to a point in S, then — by the standard removal of singularity result with embedded Lagrangian boundary conditions — all u_i extend smoothly to this point. Hence one can prescribe S to be the set of discontinuities of u_1.

In fact, the removal of singularity for squashed eights established in [Bo1, Appendix A] shows that u_0 and u_2 extend continuously to any point in S under the hypothesis that L_{01} and L_{12} have cleanly-immersed composition. In this case, the only map with any discontinuities is u_1.

At the points of energy concentration, we will see that four types of bubbles may occur: the familiar sphere and disk bubbles, and the novel figure eight and squashed eight bubbles. These novel types of bubbles result from energy concentrating on the limit seam $(-\rho,\rho)\times\{0\}$ in such a way that after rescaling (to achieve uniform gradient bounds), the middle squiggly strip converges to a straight strip of constant width, or zero width in the case of a squashed eight bubble. Note here that limit maps of this rescaling will be pseudoholomorphic with respect to the almost complex structures at the point of energy concentration.

Definition 3.2.5. Fix domain-independent almost complex structures $J_\ell \in \mathcal{J}(M_\ell,\omega_\ell)$ for $\ell = 0,1,2$.

- A **figure eight bubble between $\mathbf{L_{01}}$ and $\mathbf{L_{12}}$** is a triple of smooth maps

$$\underline{w} = \begin{pmatrix} w_0\colon \mathbb{R}\times(-\infty,-\frac{1}{2}] \to M_0 \\ w_1\colon \mathbb{R}\times[-\frac{1}{2},\frac{1}{2}] \to M_1 \\ w_2\colon \mathbb{R}\times[\frac{1}{2},\infty) \to M_2 \end{pmatrix}$$

  that satisfy the Cauchy–Riemann equations $\partial_s w_\ell + J_\ell(w_\ell)\partial_t w_\ell = 0$ for $\ell = 0,1,2$, fulfill the seam conditions

$$(w_0(s,-\tfrac{1}{2}), w_1(s,-\tfrac{1}{2})) \in L_{01}, \quad (w_1(s,\tfrac{1}{2}), w_2(s,\tfrac{1}{2})) \in L_{12} \qquad \forall\, s\in\mathbb{R},$$

  and have finite energy

$$\int w_0^*\omega_0 + \int w_1^*\omega_1 + \int w_2^*\omega_2 = \tfrac{1}{2}\Big(\int|\mathrm{d}w_0|^2 + \int|\mathrm{d}w_1|^2 + \int|\mathrm{d}w_2|^2\Big) < \infty.$$

- A **squashed eight bubble with seam in $\mathbf{L_{01}\times_{M_1}L_{12}}$** is a triple of smooth maps

$$\underline{w} = \begin{pmatrix} w_0\colon \mathbb{R}\times(-\infty,0] \to M_0 \\ w_1\colon \mathbb{R} \to M_1 \\ w_2\colon \mathbb{R}\times[0,\infty) \to M_2 \end{pmatrix}$$

  that satisfy the Cauchy–Riemann equations $\partial_s w_\ell + J_\ell(w_\ell)\partial_t w_\ell = 0$ for $\ell\in\{0,2\}$, fulfill the generalized seam condition

$$(w_0(s,0), w_1(s), w_1(s), w_2(s,0)) \in L_{01}\times_{M_1} L_{12} \qquad \forall\, s\in\mathbb{R},$$

and have finite energy

$$\textstyle\int w_0^*\omega_0 + \int w_2^*\omega_2 \;=\; \frac{1}{2}\Big(\int |\mathrm{d}w_0|^2 + \int |\mathrm{d}w_2|^2\Big) \;<\; \infty.$$

The name "figure eight" for the first type of pseudoholomorphic quilt comes from an equivalent description via stereographic projection (as explained in the following remark), while the name "squashed eight" indicates that the second type of quilt can occur as limits of figure eights whose entire energy concentrates at infinity, corresponding to shrinking the middle strip. Alternatively, squashed eights can be viewed as punctured disk bubbles $D\setminus\{1\} \to M_0^- \times M_2$ with boundary mapping to the immersed Lagrangian $L_{01} \circ L_{12}$ in such a way that it has a smooth lift to $L_{01} \times_{M_1} L_{12}$. As explained below, the singularity cannot necessarily be removed.

Remark 3.2.6.

- Recall that a pseudoholomorphic map $\mathbb{R}^2 \to M$ gives rise to a punctured pseudoholomorphic sphere $w : S^2\setminus\{(0,0,1)\} \to M$ via stereographic projection $S^2\setminus\{(0,0,1)\} \to \mathbb{R}^2$, where we identify $S^2 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 = 1\}$ with the unit sphere in $\mathbb{R}^3$. If the energy $\int w^*\omega_M$ is finite, then w extends smoothly to the puncture $(0,0,1)$ by the standard removal of singularity theorem.

- Similarly, one can view a pseudoholomorphic disk with boundary on L_{01} as a quilt on S^2 arising from a quilt on $\mathbb{R}^2$, given by a J_0-holomorphic patch $w_0 : \mathbb{R} \times (-\infty, 0] \to M_0$ and a J_1-holomorphic patch $w_1 : \mathbb{R} \times [0,\infty) \to M_1$ satisfying the seam conditions $\big(w_0(s,0), w_1(s,0)\big) \in L_{01}$, as follows: Stereographic projection lifts these to pseudoholomorphic maps $w_0 : S^2\setminus\{(0,0,1)\}\cap\{y \le 0\} \to M_0$ and $w_1 : S^2\setminus\{(0,0,1)\}\cap\{y \ge 0\} \to M_1$ defined on the two punctured hemispheres, which map the common boundary to L_{01}. The standard removal of singularity can be interpreted to say that w_0 and w_1 extend smoothly to the puncture $(0,0,1)$, thus forming a pseudoholomorphic quilted sphere with one seam — the equator $\{y = 0\}$.

  The two hemispheres are conformal to disks, so that the extended maps w_0, w_1 can be combined to a single pseudoholomorphic map from the disk to $M_0^- \times M_1$ w.r.t. the almost complex structure $(-J_0) \times J_1$, with boundary values in L_{01}.

- A squashed eight bubble gives rise to a quilt on S^2 as in the previous item, but due to the generalized nature of the seam condition, the removal of singularity is less standard. Under the hypothesis that L_{01} and L_{12} have cleanly-immersed composition, [Bo1, Appendix A] yields continuous extensions of w_0 and w_2 across $\{(0,0,1)\}$, thus giving rise to a continuous but not necessarily smooth map from the disk to $M_0^- \times M_2$ with boundary values in $L_{01} \circ L_{12}$.

- In the case of the figure eight bubble, (w_0, w_1, w_2) is a pseudoholomorphic quilt with total domain $\mathbb{R}^2$, which maps the seam $\mathbb{R} \times \{-\frac{1}{2}\}$ to L_{01} and the seam $\mathbb{R} \times \{\frac{1}{2}\}$ to L_{12}. Pulling these maps back to the sphere by stereographic projection, we obtain a pseudoholomorphic quilt whose domain is the punctured sphere, and which consists of the following patches:

$$\begin{aligned}
&w_0 \colon S^2\setminus\{(0,0,1)\} \cap \{y \le -\tfrac{1}{2}(1-z)\} \;\to\; M_0,\\
&w_1 \colon S^2\setminus\{(0,0,1)\} \cap \{-\tfrac{1}{2}(1-z) \le y \le \tfrac{1}{2}(1-z)\} \;\to\; M_1,\\
&w_2 \colon S^2\setminus\{(0,0,1)\} \cap \{\tfrac{1}{2}(1-z) \le y\} \;\to\; M_2.
\end{aligned}$$

This quilt maps the seam $\{y = -\frac{1}{2}(1-z)\}$ to L_{01} and the seam $\{y = \frac{1}{2}(1-z)\}$ to L_{12}. The union of these two seams $y = \pm\frac{1}{2}(1-z)$ on the sphere looks like the figure eight when viewed from the positive z-axis: two circles that intersect tangentially at $(0,0,1)$. [Bo1] establishes a continuous removal of singularity for (w_0, w_1, w_2) at this tangential intersection when L_{01} and L_{12} have cleanly-immersed composition.

We now turn to the definition of, and lower bounds on, the minimal bubbling energy $\hbar$, which we will need to control the number of bubbling points in the proof of Theorem 3.3.1.

Definition 3.2.7. The **minimal bubbling energy for almost complex structures $\mathbf{J_0}, \mathbf{J_1}, \mathbf{J_2}$** as in (3.5) is the minimum $\hbar := \min\{\hbar_{S^2}, \hbar_{D^2}, \hbar_{L_{01}\circ L_{12}}, \hbar_8\}$ of the following types of bubble energies.[6]

- The **minimal sphere energy** $\hbar_{S^2}$ is the minimal energy of a nonconstant[7] $J_\ell(s_0, t_0)$-holomorphic sphere in M_ℓ for any $\ell = 0, 1, 2$ and $(s_0, t_0) \in [-\rho, \rho]^2$.
- The **minimal disk energy** $\hbar_{D^2}$ is the minimal energy of a nonconstant pseudoholomorphic disk in $(M_0 \times M_1, (-J_0(s_0, 0)) \times J_1(s_0, 0))$ with boundary on L_{01} or in $(M_1 \times M_2, (-J_1(s_0, 0)) \times J_2(s_0, 0))$ with boundary on L_{12} for any $s_0 \in [-\rho, \rho]$.
- The **minimal figure eight energy** $\hbar_8$ is the minimal energy of a nonconstant $(J_0(s_0, 0), J_1(s_0, 0), J_2(s_0, 0))$-holomorphic figure eight bubble between L_{01} and L_{12} for any $s_0 \in [-\rho, \rho]$.
- The **minimal squashed eight bubble energy** $\hbar_{L_{01}\circ L_{12}}$ is the minimal energy of a nonconstant $(J_0(s_0, 0), J_2(s_0, 0))$-holomorphic squashed eight with seam in $L_{01} \times_{M_1} L_{12}$ for any $s_0 \in [-\rho, \rho]$.

In the remainder of the section, we prove two results related to the minimal figure eight energy. We begin by establishing positivity $\hbar_8 > 0$ in Lemma 3.2.8, which we will need in §3.3 to bound the number of bubbles during strip shrinking. This considerably strengthens the bubbling analysis in [WeWo1], which merely proves that the number of bubbling points must be finite.

The final result, Proposition 3.2.11, is a weak removal of singularity for any figure eight and squashed eight bubble. It applies even when the geometric composition $L_{01} \circ L_{12}$ is not immersed and yields a tuple of smooth maps with compact quilted domain that approximately capture the energy of the bubble and thus can be used in Remark 3.2.10 to give a topological understanding of the possible bubble energies.

Lemma 3.2.8. *Fix $\rho > 0$ and sequences $(J_0^\nu, J_1^\nu, J_2^\nu)_{\nu\in\mathbb{N}_0}$ of $\mathcal{C}^3$ almost complex structures on $[-\rho, \rho]^2$ as in (3.5), such that J_ℓ^ν is locally bounded in $\mathcal{C}^3$ and such that the $\mathcal{C}^2_{\mathrm{loc}}$-limit of J_ℓ^ν is a $\mathcal{C}^\infty$ almost complex structure. Then $\inf_\nu \hbar(J_0^\nu, J_1^\nu, J_2^\nu)$ is positive, where $\hbar$ is the minimum bubbling energy as in Definition 3.2.7.*

We will prove this energy bound by contradiction: Given a sequence of figure eight or squashed eight bubbles with positive energy tending to zero, we rescale to produce a nonconstant tuple of maps, which is a contradiction to the scale-invariance of energy. Here

[6] For noncompact manifolds as in Remark 3.3.4, spheres, disks, and figure eights touching or contained in the boundary strata of a compactification have to be considered here.

[7] If there are no nonconstant pseudoholomorphic spheres, e.g. because the symplectic manifolds are exact, then we set $\hbar_{S^2} = \inf\emptyset := \infty$; and similarly in the following.

the convergence of the rescaled maps will be deduced from the following result of [Bol], which establishes $\mathcal{C}^\infty$-compactness given a uniform gradient bound. It uses the notion of a **symmetric complex structure** on $[-\rho,\rho]^2$, which is a complex structure j such that the equality

$$j(s,t) = -\sigma \circ j(s,-t) \circ \sigma$$

holds for any $(s,t) \in [-\rho,\rho]^2$, where σ is the conjugation $\alpha\partial_s + \beta\partial_t \mapsto \alpha\partial_s - \beta\partial_t$. (The standard complex structure, for instance, is symmetric.)

Theorem 3.2.9 (Thm. 3.3, [Bol]). *There exists $\epsilon > 0$ such that the following holds: Fix $k \in \mathbb{N}_{\geq 1}$, positive reals $\delta^\nu \to 0$ and $\rho > 0$, symmetric complex structures j^ν on $[-\rho,\rho]^2$ that converge $\mathcal{C}^\infty$ to j^∞ with $\|j^\infty - i\|_{\mathcal{C}^0} \leq \epsilon$, and $\mathcal{C}^{k+2}$-bounded sequences of $\mathcal{C}^{k+2}$ almost complex structures J_ℓ^ν on $[-\rho,\rho]^2$ as in (3.5) such that the $\mathcal{C}^{k+1}$-limit of each (J_ℓ^ν) is a $\mathcal{C}^\infty$ almost complex structure.*

Then if $(v_0^\nu, v_1^\nu, v_2^\nu)$ is a sequence of size-(δ^ν,ρ) $(J_0^\nu, J_1^\nu, J_2^\nu, j^\nu)$-holomorphic squiggly strip quilts for (L_{01}, L_{12}) with uniformly bounded gradients,

$$\sup_{\nu\in\mathbb{N},\,(s,t)\in[-\rho,\rho]^2} |\mathrm{d}v^\nu|(s,t) < \infty,$$

then there is a subsequence in which $(v_0^\nu(t-\delta^\nu))$, $(v_1^\nu|_{t=0})$, $(v_2^\nu(t+\delta^\nu))$ converge $\mathcal{C}^k_{\mathrm{loc}}$ to a $(J_0^\infty, J_2^\infty, i)$-holomorphic size-$\rho$ degenerate strip quilt $(v_0^\infty, v_1^\infty, v_2^\infty)$ for $L_{01} \times_{M_1} L_{12}$.

If the inequality $\liminf_{\nu\to\infty,(s,t)\in[-\rho,\rho]^2} |\mathrm{d}v^\nu|(s,t) > 0$ holds, then v_0^∞, v_2^∞ are not both constant.

Proof of Lemma 3.2.8. We begin by proving energy quantization for the figure eight bubble. Suppose by contradiction that there is a sequence $\underline{w}^\nu = (w_0^\nu, w_1^\nu, w_2^\nu)$ of $(J_0^\nu(\sigma^\nu,0), J_1^\nu(\sigma^\nu,0), J_2^\nu(\sigma^\nu,0))$-holomorphic nonconstant figure eight bubbles for some $(\sigma^\nu) \subset [-\rho,\rho]$, with energy $E(\underline{w}^\nu) \to 0$. Then, despite dealing with a quilted domain, we can deduce $\lim_{\nu\to\infty} \sup_{\ell\in\{0,1,2\}} \sup |\mathrm{d}w_\ell^\nu| = 0$ from the mean value inequality $|\mathrm{d}u(z)|^2 \leq Cr^{-2}\int_{B_r(z)} |\mathrm{d}u|^2$ for pseudoholomorphic maps (see e.g. [McSa, Lemma 4.3.1] or [Wel, Theorem 1.3, Lemma A.1]).[8] Indeed, it applies to each of the maps w_0, w_1, w_2 on balls of radius $\frac{1}{2}$ that do not intersect seams, and it applies to the folded maps $(w_0(s,-\frac{1}{2}-t), w_1(s,-\frac{1}{2}+t))$ and $(w_1(s,\frac{1}{2}-t), w_2(s,\frac{1}{2}+t))$ on partial balls of radius $\frac{1}{2}$ that intersect the boundary of the domain $\mathbb{R}\times[0,1]$, where these maps are defined, only in $\mathbb{R}\times\{0\}$, where we have Lagrangian boundary conditions in L_{01} resp. L_{12}. Together, these balls cover the entire domain of the figure eight, and thus prove the uniform gradient convergence.

Next, since each triple is nonconstant we can find a subsequence (still denoted $(\underline{w}^\nu)_{\nu\in\mathbb{N}}$), an index $\ell_0 \in \{0,1,2\}$, and points (s^ν, t^ν) in the domain of $w_{\ell_0}^\nu$ such that $\delta^\nu := |\mathrm{d}w_{\ell_0}^\nu(s^\nu,t^\nu)| \geq \frac{1}{2}\sup_{\ell\in\{0,1,2\}} \sup |\mathrm{d}w_\ell^\nu|$. We just showed $\delta^\nu \to 0$, and we claim that in fact $\delta^\nu t^\nu \to 0$. Indeed, this only requires a proof in the case $|t^\nu| \to \infty$. In that case we may apply the mean value inequality on balls of radius $t^\nu - 1$ to obtain $\delta^\nu t^\nu \to 0$. By shifting each triple of maps in the s-direction, we may moreover assume $s^\nu = 0$ for all $\nu \in \mathbb{N}$.

[8] Here we use the metric on M_ℓ that is induced by ω_ℓ and J_ℓ^∞ by (3.6). Note that for any fixed ν, the convergence $J_\ell^\nu \to J_\ell^\infty$ implies that the norm induced by this metric is equivalent to the norm induced by ω_ℓ and J_ℓ^ν; furthermore, the constant of equivalence can be chosen to be independent of ν. This in particular yields uniform constants in the mean value inequalities.

Now rescale $v_\ell^\nu(s,t) := w_\ell^\nu(s/\delta^\nu, t/\delta^\nu)$ to obtain maps

$$v_0^\nu : \mathbb{R} \times (-\infty, \tfrac{1}{2}\delta^\nu] \to M_0, \qquad v_1^\nu : \mathbb{R} \times [-\tfrac{1}{2}\delta^\nu, \tfrac{1}{2}\delta^\nu] \to M_1, \qquad v_2^\nu : \mathbb{R} \times [\tfrac{1}{2}\delta^\nu, \infty) \to M_2.$$

These maps are $J_\ell(\sigma^\nu, 0)$-holomorphic and satisfy the following seam conditions:

$$(v_0^\nu(s, -\tfrac{1}{2}\delta^\nu), v_1^\nu(s, -\tfrac{1}{2}\delta^\nu)) \in L_{01}, \qquad (v_1^\nu(s, \tfrac{1}{2}\delta^\nu), v_2^\nu(s, \tfrac{1}{2}\delta^\nu)) \in L_{12} \qquad \forall\, s \in \mathbb{R}.$$

The rescaling was chosen to ensure an upper bound on the gradient, $\sup_{\ell\in\{0,1,2\}} |\mathrm{d}v_\ell^\nu| \leq 2$, as well as a lower bound $|\mathrm{d}v_{\ell_0}^\nu(0,\tau^\nu)| \geq 1$ for $\tau^\nu := \delta^\nu t^\nu \to 0$. Theorem 3.2.9 implies that the restrictions of $v_0^\nu(s, t - \frac{1}{2}\delta^\nu)$ resp. $v_2^\nu(s, t + \frac{1}{2}\delta^\nu)$ to $(-1,1) \times (-1,0]$ resp. $(-1,1) \times [0,1)$ converge $\mathcal{C}^1_{\mathrm{loc}}$ to maps v_0^∞ resp. v_2^∞, and that at least one of the limit maps is nonconstant. This is in contradiction to the scale-invariant energy converging to 0:

$$\begin{aligned}
0 &< \int_{(-1,1)\times(-1,0]} (v_0^\infty)^*\omega_0 + \int_{(-1,1)\times[0,1)} (v_2^\infty)^*\omega_2 \\
&= \lim_{\nu\to\infty} \left(\int_{(-1,1)\times[0,1)} (v_0^\nu)^*\omega_0 + \int_{(-1,1)\times[0,1)} (v_2^\nu)^*\omega_2 \right) \\
&= \lim_{\nu\to\infty} \left(\int_{B_{(-\delta^\nu,\delta^\nu)\times[0,\delta^\nu)}} (w_0^\nu)^*\omega_0 + \int_{B_{(-\delta^\nu,\delta^\nu)\times[0,\delta^\nu)}} (w_2^\nu)^*\omega_2 \right) \leq \liminf_{\nu\to\infty} E(\underline{w}^\nu) = 0.
\end{aligned}$$

Hence we have proven the existence of a positive lower bound $\hbar_8 > 0$.

A similar argument establishes energy quantization for squashed eights. One difference between the two arguments is that the mean value inequality as stated in the literature requires the boundary to map to an *embedded* Lagrangian, so we cannot deduce uniform gradient convergence to zero. Hence we consider two cases, depending on whether the limit $L := \lim_{\nu\to\infty} \sup_{\ell\in\{0,1,2\}} \sup |\mathrm{d}w_\ell^\nu|$ (which exists after passing to a subsequence) is finite or infinite.

- $\mathbf{L} \in [\mathbf{0}, \infty)$: Center and rescale as in the proof of $\hbar_8 > 0$. To deal with the immersed boundary condition, choose a finite open cover $L_{01} \times_{M_1} L_{12} = \bigcup_{i=1}^N U_i$ such that $\pi_{02} : L_{01} \times_{M_1} L_{12} \to M_0^- \times M_2$ restricts to an embedding on each U_i. Since the rescaled maps have uniformly-bounded gradient, and since their boundary values have smooth lifts to $L_{01} \times_{M_1} L_{12}$, we can pass to a subsequence and bounded domain to work with embedded boundary conditions in some $\pi_{02}(U_i)$. Depending on whether L is finite or infinite, we can then appeal to either standard bootstrapping techniques (e.g. [McSa, Theorem 4.1.1]) or Theorem 3.2.9 to obtain convergence and hence a contradiction.

- $\mathbf{L} = \infty$: Choose points (s^ν, t^ν) and ℓ_0 so that $|\mathrm{d}w_{\ell_0}^\nu(s^\nu, t^\nu)| \to \infty$. As in the proof of Theorem 3.3.1, we can apply the Hofer trick to vary the points (s^ν, t^ν) slightly and produce numbers R^ν, ϵ^ν; rescaling by $v_\ell^\nu(s,t) := w_\ell^\nu(s^\nu + s/R^\nu, t^\nu + t/R^\nu)$ produces a sequence of maps which has nonconstant limit.

Finally, $\hbar > 0$ follows from the above since we have standard lower bounds $\hbar_{S^2}, \hbar_{D^2} > 0$, which can be proven by a single mean value inequality applied to balls resp. half balls of large radius, see e.g. [McSa, Proposition 4.1.4]. □

Remark 3.2.10. The minimal bubbling energies $\hbar_{S^2}, \hbar_{D^2}, \hbar_8, \hbar_{L_{01}\circ L_{12}}$ in Definition 3.2.7 can also be bounded below by concrete topological quantities

$$\hbar_{S^2} \geq \hbar_{S^2}^{\mathrm{top}}, \qquad \hbar_{D^2} \geq \hbar_{D^2}^{\mathrm{top}}, \qquad \min\{\hbar_8, \hbar_{L_{01}\circ L_{12}}\} \geq \hbar_8^{\mathrm{top}}$$

rather than the abstract analytic lower bound from Lemma 3.2.8. For sphere and disk bubbles, this topological quantity is the minimal positive symplectic area of spherical or disk (relative) homotopy classes. Proposition 3.2.11 bounds the minimal energy of squashed eight and figure eight bubbles by the minimal positive symplectic area of "quilted homotopy classes"

$$\hbar_8^{\mathrm{top}} := \inf\left\{ \sum_{\ell\in\{0,1,2\}} \langle[\omega_\ell],[u_\ell]\rangle \;\middle|\; \begin{array}{c} u_0 : D^2 \to M_0, \\ u_1 : [0,2\pi]\times[-\frac12,\frac12] \to M_1, \\ u_2 : D^2 \to M_2, \end{array} \quad (3.11),(3.12) \right\}$$

for which u_0, u_1, u_2 are continuous and satisfy the seam conditions

$$(u_0(e^{-i\theta}), u_1(\theta,-\tfrac12)) \in L_{01}, \quad (u_1(\theta,\tfrac12), u_2(e^{i\theta})) \in L_{12} \qquad \forall\, \theta\in[0,2\pi] \tag{3.11}$$

and the "constant limit conditions"

$$u_1(0,t_1) = u_1(0,t_2), \quad u_1(2\pi,t_1) = u_1(2\pi,t_2) \quad \forall\, t_1,t_2 \in [-\tfrac12,\tfrac12]. \tag{3.12}$$

We differentiate such quilt maps by the relation between the two constants $u_1(0,-)$ and $u_1(2\pi,-)$:

- A **(non-switching) homotopy figure eight** is a tuple of maps (u_0,u_1,u_2) as above with $u_1(2\pi,-) = u_1(0,-)$. That is, $u_1 : S^1\times[-\frac12,\frac12] \to M_1$ is in fact defined on an annulus with seam conditions that identify $S^1\times\{\pm\frac12\}$ with the boundaries of the two disk patches.
- A **sheet-switching homotopy figure eight** is a tuple of maps (u_0,u_1,u_2) as above with $u_1(0,-) =: u_1^- \neq u_1^+ := u_1(2\pi,-)$, where u_1^-, u_1^+ represent two different lifts of $\big(u_0(1),u_2(1)\big) \in L_{01}\circ L_{12}$ to $L_{01}\times_{M_1} L_{12}$.

Note that sphere homotopy classes as well as disk homotopy classes for L_{01} and L_{12} can be represented by non-switching homotopy figure eights with one or two constant patches.[9] However, $\hbar^{\mathrm{top}} := \min\{\hbar_{S^2}^{\mathrm{top}}, \hbar_{D^2}^{\mathrm{top}}, \hbar_8^{\mathrm{top}}\}$ is not generally positive unless the symplectic and Lagrangian manifolds have very simple topology. For example, we have $\hbar_{S^2}^{\mathrm{top}} = 0$ as soon as $\langle[\omega_\ell],\pi_2(M_\ell)\rangle \subset \mathbb{R}$ contains two incommensurate values for some $\ell\in\{0,1,2\}$.

The possible homotopy classes of figure eight bubbles in the above remark can be deduced from the removal of singularity theorem in [Bo1]. However, this also follows from the following weaker result which requires fewer estimates. It yields not a pseudoholomorphic quilt on S^2 but a smooth quilt map with domain $S^2 \cong (D^2)^- \cup (S^1\times[0,1]) \cup D^2$ that approximately captures the energy of the bubble. This result was first announced in [We2], but we include it here for convenience. It is the only point in this chapter where we will not assume L_{01} and L_{12} to have immersed composition.

[9] Given a sphere or disk bubble, we can attach it to a constant homotopy figure eight under mild hypotheses (e.g. that L_{01}, L_{12} are nonempty and M_1 is connected): Given any two points on L_{01} and L_{12}, make a zero-energy homotopy figure eight with these values on the seams, u_0 and u_2 constant, and u_1 a path between the two projections.

Proposition 3.2.11. *Let $L_{01} \subset M_0^- \times M_1$, $L_{12} \subset M_1^- \times M_2$ be compact Lagrangian correspondences, and let (w_0, w_1, w_2) be either (1) a figure eight bubble between L_{01} and L_{12} or (2) a squashed eight bubble with seam in $L_{01} \times_{M_1} L_{12}$, where w_1 is the pullback $w_1(s,t) := \overline{w}_1(s)$. Then for any $\epsilon > 0$ there exist smooth maps $u_0 : D^2 \to M_0$, $\widehat{u}_1 : [0, 2\pi] \times [-\frac{1}{2}, \frac{1}{2}] \to M_1$, $u_2 : D^2 \to M_2$ satisfying the seam conditions*

$$\big(u_0(e^{-i\theta}), \widehat{u}_1(\theta, -\tfrac{1}{2})\big) \in L_{01}, \quad \big(\widehat{u}_1(\theta, \tfrac{1}{2}), u_2(e^{i\theta})\big) \in L_{12} \qquad \forall\, e^{i\theta} \in \partial D^2 \cong \mathbb{R}/2\pi\mathbb{Z},$$

and whose energy is ϵ-close to that of (w_0, w_1, w_2),

$$\Big| \Big(\textstyle\int u_0^*\omega_0 + \int \widehat{u}_1^*\omega_1 + \int u_2^*\omega_2 \Big) - \Big(\int w_0^*\omega_0 + \int w_1^*\omega_1 + \int w_2^*\omega_2 \Big) \Big| \le \epsilon.$$

Moreover, $\widehat{u}_1$ is constant on the two lifts of the line $\{[0] = [2\pi]\} \times [-\frac{1}{2}, \frac{1}{2}] \subset S^1 \times [-\frac{1}{2}, \frac{1}{2}]$, so that $\widehat{u}_1|_{\{0\}\times[-\frac{1}{2},\frac{1}{2}]} \equiv p_1^+$, $\widehat{u}_1|_{\{2\pi\}\times[-\frac{1}{2},\frac{1}{2}]} \equiv p_1^-$ form together with $p_0 := u_0(e^{i0})$, $p_2 := u_2(e^{i0})$ two lifts $(p_0, p_1^{\pm}, p_1^{\pm}, p_2) \in L_{01} \times_{M_1} L_{12}$ of the same point $(p_0, p_2) \in L_{01} \circ L_{12}$.

In particular, if $\pi_{02} : L_{01} \times_{M_1} L_{12} \to L_{01} \circ L_{12}$ is injective, then $\widehat{u}_1$ can be chosen such that it induces a smooth map $u_1 : S^1 \times [-\frac{1}{2}, \frac{1}{2}] \to M_1$.

For the proof of Proposition 3.2.11, we will need an extension result. To state it we will use the following notation for partitioning the closed unit ball $\overline{B}_1(0) \subset \mathbb{R}^2$ into four quadrants:

$$\begin{aligned} U_0 &:= \{(x,y) \in \overline{B}(0,1) \mid y \le x,\ y \le -x\}, \qquad & U_1 &:= \{(x,y) \in \overline{B}(0,1) \mid x \ge y,\ x \ge -y\}, \\ U_2 &:= \{(x,y) \in \overline{B}(0,1) \mid y \ge x,\ y \ge -x\}, \qquad & U_3 &:= \{(x,y) \in \overline{B}(0,1) \mid x \le y,\ x \le -y\}. \end{aligned} \tag{3.13}$$

The resulting partition of the boundary circle $\partial\overline{B}_1(0)$ will be denoted by $A_i := U_i \cap \partial\overline{B}_1(0)$ for $i = 0, 1, 2, 3$, and we denote the intersections of these arcs by $p_{i(i+1)} := A_i \cap A_{i+1}$ for i mod 4. We denote the length of a path $\sigma_i : A_i \to X_i$ with respect to g_i by $\ell(\sigma_i) := \int_{A_i} |\mathrm{d}\sigma_i|$.

Lemma 3.2.12. *Let (X_i, g_i) be Riemannian manifolds equipped with closed 2-forms ω_i for $i = 0, 1, 2$, and let $Y_{01} \subset X_0 \times X_1$, $Y_{12} \subset X_1 \times X_2$ be closed submanifolds. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that the following extension property holds: Suppose that $\sigma_i : A_i \to X_i$ for $i = 0, 1, 2, 3$ are smooth arcs that satisfy*

$$\ell(\sigma_i) \le \delta, \qquad (\sigma_i(p_{i(i+1)}), \sigma_{i+1}(p_{i(i+1)})) \in Y_{i(i+1)} \qquad \forall\, i \bmod 4. \tag{3.14}$$

Here we denote $X_3 := X_1$, $Y_{23} := Y_{12}^T$, and $Y_{30} := Y_{01}^T$, with $(\cdot)^T$ denoting the interchange of factors in $X_i \times X_{i+1}$. Then there exist smooth extensions $\widetilde{\sigma}_i : U_i \to X$ of $\widetilde{\sigma}_i|_{A_i} = \sigma_i$ such that

$$\int_{U_i} \widetilde{\sigma}_i^*\omega_i \le \epsilon, \qquad (\widetilde{\sigma}_i(p), \widetilde{\sigma}_{i+1}(p)) \in Y_{i(i+1)} \quad \forall\, p \in U_i \cap U_{i+1} \qquad \forall\, i \bmod 4.$$

Proof of Lemma 3.2.12. Set $a_i := \sigma_i(p_{(i-1)i})$, $b_i := \sigma_i(p_{i(i+1)})$ for i mod 4. For a constant $\epsilon' > 0$ that we will fix later in the proof, let us show that if δ is chosen small enough, there exist $x^{\pm} = (x_0, x_1^{\pm}, x_1^{\pm}, x_2) \in Y_{01} \times_{X_1} Y_{12}$ (two lifts of the same point in $Y_{01} \circ Y_{12} \subset X_0 \times X_2$) such that the following distances with respect to the metric $g_0 \oplus g_1 \oplus g_1 \oplus g_2$ are bounded,

$$\max\big\{ d((b_0, a_1, b_1, a_2), x^+),\ d((a_0, b_3, a_3, b_2), x^-) \big\} \le \epsilon'. \tag{3.15}$$

Suppose by contradiction that the sequences $(\sigma_0^\nu, \sigma_1^\nu, \sigma_2^\nu, \sigma_3^\nu)$ satisfy (3.14) for a sequence $\delta^\nu \to 0$ but

$$\min_{x^\pm=(x_0,x_1^\pm,x_1^\pm,x_2)\in Y_{01}\times_{X_1}Y_{12}} \max\bigl\{d((b_0^\nu, a_1^\nu, b_1^\nu, a_2^\nu), x^+)\,,\, d((a_0^\nu, b_3^\nu, a_3^\nu, b_2^\nu), x^-)\bigr\} > \epsilon' \quad (3.16)$$

for all $\nu \in \mathbb{N}$ with $a_i^\nu := \sigma_i^\nu(p_{(i-1)i})$, $b_i^\nu := \sigma_i^\nu(p_{i(i+1)})$. Since $(b_0^\nu, a_1^\nu) \in Y_{01}, (b_1^\nu, a_2^\nu) \in Y_{12}, (a_3^\nu, b_2^\nu) \in Y_{12}, (a_0^\nu, b_3^\nu) \in Y_{01}$ and Y_{01}, Y_{12} are compact, we may pass to a subsequence and assume that a_i^ν, b_i^ν have limits a_i^∞, b_i^∞ as $\nu \to \infty$. These limits have to coincide $a_i^\infty = b_i^\infty$ since they are the limits of endpoints of the paths σ_i^ν whose length $\ell(\sigma_i^\nu) \le \delta^\nu$ goes to zero with $\nu \to \infty$. This gives rise to two lifts $x^+ := (a_0^\infty, a_1^\infty, a_1^\infty, a_2^\infty), x^- := (a_0^\infty, a_3^\infty, a_3^\infty, a_2^\infty) \in Y_{01} \times_{X_1} Y_{12}$ since $(a_0^\infty, a_1^\infty) = \lim_{\nu\to\infty}(b_0^\nu, a_1^\nu)$ and $(a_0^\infty, a_3^\infty) = \lim_{\nu\to\infty}(a_0^\nu, b_3^\nu)$ are limits in the closed submanifold Y_{01} and $(a_1^\infty, a_2^\infty) = \lim_{\nu\to\infty}(b_1^\nu, a_2^\nu)$ and $(a_3^\infty, a_2^\infty) = \lim_{\nu\to\infty}(a_3^\nu, b_2^\nu)$ are limits in the closed submanifold Y_{12}, and they contradict (3.16) since both distances converge to zero, e.g.

$$d((b_0^\nu, a_1^\nu, b_1^\nu, a_2^\nu), x^+) \le d(b_0^\nu, a_0^\infty = b_0^\infty) + d(a_1^\nu, a_1^\infty) + d(b_1^\nu, a_1^\infty = b_1^\infty) + d(a_2^\nu, a_2^\infty) \underset{\nu\to\infty}{\longrightarrow} 0.$$

With that we may assume to have lifts $x^\pm = (x_0, x_1^\pm, x_1^\pm, x_2) \in Y_{01} \times_{X_1} Y_{12}$ satisfying (3.15) and begin to construct the extensions $\widetilde{\sigma}_i$ by

$$\widetilde{\sigma}_0(0) := x_0, \qquad \widetilde{\sigma}_1(0) := x_1^+, \qquad \widetilde{\sigma}_2(0) := x_2, \qquad \widetilde{\sigma}_3(0) := x_1^-.$$

To construct $\widetilde{\sigma}_i \times \widetilde{\sigma}_{i+1} : U_i \cap U_{i+1} \to Y_{i(i+1)}$, note that the given values on both ends of this line segment are at distance at most ϵ' in $Y_{i(i+1)}$. Hence for sufficiently small ϵ' we may use local charts of the submanifolds Y_{01}, Y_{12} to choose each extension $\widetilde{\sigma}_i : \partial U_i \to X_i$ of $\widetilde{\sigma}_i|_{A_i} = \sigma_i$ such that they satisfy the seam conditions $(\widetilde{\sigma}_i(p), \widetilde{\sigma}_{i+1}(p)) \in Y_{i(i+1)}$ and length bound $\ell(\widetilde{\sigma}_i|_{\partial U_i}) \le 2\epsilon' + \delta$. By choosing ϵ' and δ sufficiently small, we can moreover ensure that each of these loops lies in contractible charts of X_i. On the one hand, that allows us to extend the given $\widetilde{\sigma}_i : \partial U_i \to X$ to a smooth map $\widetilde{\sigma}_i : U_i \to X_i$. On the other hand, in each such contractible chart $V \subset X_i$ the given 2-form $\omega_i|_V = \mathrm{d}\eta_V$ has a uniformly bounded primitive $\eta_V \in \Omega^1(V)$, which gives the desired bound

$$\int_{U_i} \widetilde{\sigma}_i^*\omega_i \;=\; \int_{\partial U_i} \widetilde{\sigma}_i^*\eta_V \;\le\; \|\eta_V\|_\infty \ell(\widetilde{\sigma}_i|_{\partial U_i}) \;\le\; \|\eta_V\|_\infty(2\epsilon' + \delta) \;\le\; \epsilon$$

for sufficiently small $\delta > 0$. In fact, we can cover the projections of the compact Lagrangians to the factors X_i with finitely many contractible charts V so that $\|\eta_V\|_\infty$ is uniformly bounded. This ensures that the choice of sufficiently small $\delta > 0$ for given $\epsilon > 0$ is independent of the arcs σ_i. □

Proof of Proposition 3.2.11. To simplify notation we shift domains so that w_0 and w_2 in case (i) as well as (ii) are parametrized by $\mathbb{R} \times (-\infty, 0]$ and $\mathbb{R} \times [0, \infty)$, respectively. On these domains we rewrite the figure eight energy integral in polar coordinates $\widetilde{w}_\ell(r, \theta) =$

$w_\ell(r\cos\theta, r\sin\theta)$ to obtain

$$
\begin{aligned}
E &:= \textstyle\int w_0^*\omega_0 + \int w_1^*\omega_1 + \int w_2^*\omega_2 \\
&= \int_{\mathbb{R}\times(-\infty,0]} r^{-2}|\partial_\theta\widetilde{w}_0|^2 \mathrm{d}s\mathrm{d}t + \int_{\mathbb{R}\times[-\frac12,\frac12]} |\partial_t w_1|^2 \mathrm{d}s\mathrm{d}t + \int_{\mathbb{R}\times[0,\infty)} r^{-2}|\partial_\theta\widetilde{w}_2|^2 \mathrm{d}s\mathrm{d}t \\
&= \lim_{R\to\infty}\int_0^R r^{-1}A(r)\,\mathrm{d}r
\end{aligned}
$$

with integrand

$$
\begin{aligned}
A(r) := \int_\pi^{2\pi} |\partial_\theta\widetilde{w}_0(r,\theta)|^2\mathrm{d}\theta + \int_0^\pi |\partial_\theta\widetilde{w}_2(r,\theta)|^2\mathrm{d}\theta + \int_{-\frac12}^{\frac12} r|\partial_t w_1(-r,t)|^2\mathrm{d}t \\
+ \int_{-\frac12}^{\frac12} r|\partial_t w_1(r,t)|^2\mathrm{d}t.
\end{aligned}
$$

The same holds for squashed eights if we drop the terms involving w_1. By assumption $\int_0^R r^{-1}A(r)\mathrm{d}r$ converges as $R\to\infty$ although $A(r)\geq 0$ and $\int_0^R r^{-1}\mathrm{d}r \to \infty$ as $R\to\infty$. Hence there exists a sequence $r_i\to\infty$ such that $A(r_i)\to 0$. Depending on a $\delta>0$ to be determined and the $\epsilon>0$ given, we now choose $r_0>1$ sufficiently large such that $A(r_0)\leq\delta$ and $\left|E-\int_0^{r_0} r^{-1}A(r)\mathrm{d}r\right|\leq \frac12\epsilon$. Denoting by $B^\pm_{r_0}$ the ball of radius r_0 around the origin in the halfplanes $\mathbb{H}^+=\mathbb{R}\times[0,\infty)$ resp. $\mathbb{H}^-=\mathbb{R}\times(-\infty,0]$, we now have approximated the energy

$$
\left|E-\left(\int_{B^-_{r_0}} w_0^*\omega_0 + \int_{[-r_0,r_0]\times[-\frac12,\frac12]} w_1^*\omega_1 + \int_{B^+_{r_0}} w_2^*\omega_2\right)\right| \leq \tfrac12\epsilon
$$

and bounded lengths of arcs

$$
\ell\big(w_1|_{\{\pm r_0\}\times[-\frac12,\frac12]}\big) \leq \sqrt{\delta/r_0}, \qquad \ell\big(\widetilde{w}_\ell|_{|z|=r_0}\big)\leq\sqrt{\pi\delta} \quad \text{for } \ell\in\{0,2\}.
$$

Here the latter for $\widetilde{w}_0$ (and analogously for $\widetilde{w}_2$ and w_1) follows from the estimate

$$
\ell\big(\widetilde{w}_0|_{|z|=r_0}\big) \leq \int_\pi^{2\pi} |\partial_\theta\widetilde{w}_0(r_0,\theta)|\mathrm{d}\theta \leq \sqrt{\pi A(r_0)}.
$$

Then for sufficiently large $r_0>1$ and small $\delta>0$, the maps $(u_0,\widehat{u}_1,u_2)$ will be constructed as extensions of $(w_0|_{B^-_{r_0}}, w_1|_{[-r_0,r_0]\times[-\frac12,\frac12]}, w_2|_{B^+_{r_0}})$. We first pull them back to the quilted sphere by stereographic projection as in Remark 3.2.6, to define a quilted map $(v_0,\widehat{v}_1,v_2)$ on the complement of a neighborhood $N\subset S^2$ of the puncture $(0,0,1)$. Thus N is a slightly-deformed ball with diameter of order r_0^{-1}, which we identify with $\overline{B}_1(0)$ in Lemma 3.2.12 so that the arcs $\sigma_0=v_0|_{\partial N}$ and $\sigma_2=v_0|_{\partial N}$ are reparametrizations of the short paths $(w_0|_{B^-_{r_0}}, w_2|_{B^+_{r_0}})$, and σ_1,σ_3 are the two connected components of $v_1|_{\partial N}$, given by reparametrizations of the short paths $w_1|_{[-r_0,r_0]\times[-\frac12,\frac12]}$. For sufficiently small $\delta>0$, Lemma 3.2.12 then provides a smooth extension of $(v_0,\widehat{v}_1,v_2)|_{\partial N}$ by a quilted map $(\widetilde{\sigma}_0,\widetilde{\sigma}_1,\widetilde{\sigma}_3,\widetilde{\sigma}_2)$ on N with total symplectic area bounded by $\frac12\epsilon$. After smoothing these extensions near ∂N, we finally construct $(u_0,\widehat{u}_1,u_2)$ by pullback of these extended maps. More precisely, we construct u_0 (and similarly u_2) by precomposition of $v_0,\widetilde{\sigma}_0$ with a smooth bijection from D^2 to $\mathrm{dom}(v_0) \cup U_0$ which maps $1 \in \partial D^2$ to the corner of U_0. (This is not possible by a diffeomorphism, but there is a smooth map with vanishing derivatives at 1.) To construct $\widehat{u}_1 : [0, 2\pi] \times [-\frac{1}{2}, \frac{1}{2}] \to M_1$ we pull back $v_1, \widetilde{\sigma}_1, \widetilde{\sigma}_3$ by a smooth map from $[0, 2\pi] \times [-\frac{1}{2}, \frac{1}{2}]$ to $\mathrm{dom}(u_1) \cup U_1 \cup U_3$ which on the boundary components $[0, 2\pi] \times \{\pm\frac{1}{2}\}$ (in polar coordinates) coincides with the bijections from ∂D^2 to $U_0 \cap (U_1 \cup U_3)$ resp. $U_2 \cap (U_1 \cup U_3)$ used in the construction of u_0, u_2, thus guaranteeing the seam conditions. It can moreover be chosen as bijection with the exception of mapping the two edges $\{0, 2\pi\} \times [-\frac{1}{2}, \frac{1}{2}]$ to the common corner point $U_1 \cap U_3$. Smoothness of these maps guarantees smoothness of the pullbacks $(u_0, \widehat{u}_1, u_2)$, and bijectivity on the complement of a zero set guarantees that they have the same symplectic area as the extension of $(v_0, \widehat{v}_1, v_2)|_{\partial N}$. Finally, $\widehat{u}_1$ by construction is constant equal to $\widetilde{\sigma}_1(0)$ on $\{0\} \times [-\frac{1}{2}, \frac{1}{2}]$ and equal to $\widetilde{\sigma}_3(0)$ on $\{2\pi\} \times [-\frac{1}{2}, \frac{1}{2}]$, and extends smoothly to an annulus if $\widetilde{\sigma}_1(0) = \widetilde{\sigma}_3(0)$. The latter is guaranteed by the seam conditions on the extensions $\widetilde{\sigma}_i$ if $\pi_{02} : L_{01} \times_{X_1} L_{12} \to L_{01} \circ L_{12}$ is injective. □

Remark 3.2.13. Under the hypothesis that L_{01}, L_{12} have immersed composition, Proposition 3.2.11 can be modified to show that a squashed eight can be approximated by a homotopy squashed eight, rather than a homotopy figure eight. In this situation, the minimum squashed eight energy $\hbar_{L_{01} \circ L_{12}}$ can be bounded below by the minimum positive symplectic area of "homotopy squashed eights":

$$\hbar^{\mathrm{top}}_{L_{01} \circ L_{12}} := \inf\bigl\{ \langle [(-\omega_0) \oplus \omega_2], [u] \rangle > 0 \,\big|\, u \in \mathcal{C}^0(D, M_0^- \times M_2), u(\partial D) \subset L_{01} \circ L_{12} \bigr\}.$$

3.3 Toward Gromov compactness for strip shrinking

In this section we state and prove the Gromov Compactness Theorem 3.3.1, which is the main result of this chapter. In order to focus on the relevant effects, rather than deal with complicated notation, Theorem 3.3.1 is stated in the setting of squiggly strip quilts, with the width of the middle strip shrinking obediently to zero. However, the results of this section directly generalize to a sequence of pseudoholomorphic quilt maps whose domains are quilted surfaces which vary only by the width of one patch — diffeomorphic to a strip or annulus — going to zero.

Theorem 3.3.1 is a refinement and generalization of [WeWo1, Theorem 3.3.1 and Lemma 3.3.2], where compactness up to energy concentration is proven for strip shrinking in the special case of embedded composition, though only in an $H^2 \cap W^{1,4}$-topology and with a lower bound on the energy concentration that has no geometric interpretation but arises by contradiction from mean value inequalities. (In fact, the $H^2 \cap W^{1,4}$-convergence does not even suffice to deduce nontriviality of the weak limit of rescaled solutions near a bubbling point.) We establish full $\mathcal{C}^\infty_{\mathrm{loc}}$-convergence in the most general natural case, with the further generalization to noncompact manifolds being discussed in Remark 3.3.4. The proof will moreover illuminate the origin of the different bubbling phenomena. Analytically, it relies on Theorem 3.2.9, a result from [Bo1]. A further generalization is that we will allow the two seams bordering the middle strip to not be straight, so that Theorem 3.3.1 allows the first author to establish a removal of singularity theorem for figure eight bubbles in [Bo1].

Theorem 3.3.1. *Fix $\rho > 0$, sequences $J_0^\nu, J_1^\nu, J_2^\nu$ of smooth almost complex structures on $[-\rho, \rho]^2$ as in* (3.5) *that converge $\mathcal{C}^\infty_{\mathrm{loc}}$ to $J_\ell^\infty \colon [-\rho, \rho]^2 \to \mathcal{J}(M_\ell, \omega_\ell)$ for $\ell = 0, 1, 2$, and a sequence $(f^\nu \colon [-\rho, \rho] \to (0, \rho/2])$ of real-analytic functions shrinking obediently to zero as in Definition 3.2.2. Then for any sequence $(\underline{v}^\nu)_{\nu \in \mathbb{N}}$ of $(J_0^\nu, J_1^\nu, J_2^\nu)$-holomorphic size-$(f^\nu, \rho)$ squiggly strip quilts for (L_{01}, L_{12}) as in Definition 3.2.1 with bounded energy $E :=$*

$\sup_{\nu\in\mathbb{N}} E(\underline{v}^\nu) < \infty$ *there exist finitely many blow-up points* $Z = \{z_1, \ldots, z_N\} \subset (-\rho,\rho)^2$ *and a subsequence that Gromov-converges in the following sense:*

1. *There exists a* (J_0^∞, J_2^∞)*-holomorphic degenerate strip quilt* $\underline{v}^\infty$ *for* $L_{01} \times_{M_1} L_{12}$ *with singularities, whose singular set* $S \subset (-\rho,\rho) \cong (-\rho,\rho)\times\{0\}$ *is contained in* $\{z_1,\ldots,z_N\} \cap (-\rho,\rho)\times\{0\}$*, such that* $(v_0^\nu(s,t-f^\nu(s)))$ *resp.* $(v_1^\nu(s,0))$ *resp.* $(v_2^\nu(s,t+f^\nu(s)))$ *converge* $\mathcal{C}^\infty_{\mathrm{loc}}$ *on the domains* $(-\rho,\rho)\times(-\rho,0]\smallsetminus Z$ *resp.* $(-\rho,\rho)\times\{0\}\smallsetminus Z$ *resp.* $(-\rho,\rho)\times[0,\rho)\smallsetminus Z$ *to* v_0^∞ *resp.* v_1^∞ *resp.* v_2^∞*.*

2. *There is a concentration of energy* $\hbar > 0$*, given by the minimal bubbling energy from Definition 3.2.7, at each* z_j *in the sense that there is a sequence of radii* $r^\nu \to 0$ *such that*

$$\liminf_{\nu\to\infty} \int_{B_{r^\nu}(z_j)} \tfrac{1}{2}|\mathrm{d}\underline{v}^\nu|^2 \;\geq\; \hbar > 0,$$

where the energy densities $|\mathrm{d}\underline{v}^\nu|$ *are defined as in* (3.10).

3. *At least one type of bubble forms near each blow-up point* $z_j = (s_j,t_j)$*: There is a sequence* (w^ν) *of (tuples of) maps obtained by rescaling the maps defined on the intersection of the respective domain with* $B_{r^\nu}(z_j)$*, which converges in* $\mathcal{C}^\infty_{\mathrm{loc}}$ *to one of the following:*

(S0),(S1),(S2): *a* $J_\ell^\infty(z_j)$*-holomorphic map* $w_\ell^\infty\colon \mathbb{R}^2 \to M_\ell$ *for* $\ell = 0,1,2$*, which can be completed to a nonconstant pseudoholomorphic sphere* $\overline{w}_\ell^\infty\colon S^2 \to M_\ell$*;*

(D01): *a* $(-J_0^\infty(s_j,0))\times J_1^\infty(s_j,0)$*-holomorphic map* $w_{01}^\infty\colon \mathbb{H} \to M_0^-\times M_1$ *with* $w_{01}^\infty(\partial\mathbb{H}) \subset L_{01}$*, which can be extended to a nonconstant pseudoholomorphic disk* $\overline{w}_{01}^\infty\colon (D,\partial D) \to (M_0^- \times M_1, L_{01})$*;*

(D12): *a* $(-J_1^\infty(s_j,0))\times J_2^\infty(s_j,0)$*-holomorphic map* $w_{12}^\infty\colon \mathbb{H} \to M_1^-\times M_2$ *with* $w_{12}^\infty(\partial\mathbb{H}) \subset L_{12}$*, which can be extended to a nonconstant pseudoholomorphic disk* $\overline{w}_{12}^\infty\colon (D,\partial D) \to (M_1^- \times M_2, L_{12})$*;*

(E012): *a nonconstant* $(J_0^\infty(s_j,0), J_1^\infty(s_j,0), J_2^\infty(s_j,0))$*-holomorphic figure eight bubble between* L_{01} *and* L_{12}*, as in Definition 3.2.5;*

(D02): *a nonconstant* $(J_0^\infty(s_j,0), J_2^\infty(s_j,0))$*-holomorphic squashed eight bubble with generalized boundary conditions in* $L_{01}\circ L_{12}$*, as in Definition 3.2.5.*

Remark 3.3.2. If the composition $L_{01}\circ L_{12}$ is cleanly immersed, then [Bo1, Thm. 2.2] guarantees a continuous removal of singularity for figure eight and squashed eight bubbles, in particular for the bubbles produced in cases (E012) and (D02) of Theorem 3.3.1. This allows us to partially characterize the singular set $S \subset \mathbb{R}$ in (1) at which $\overline{v}_1$ does not extend continuously, and hence to which v_0, v_2 may not extend smoothly: A necessary condition for a bubbling point z_j to lie in S is that a sheet-switching bubble can be found by rescaling near z_j, i.e. a squashed eight bubble whose boundary arc on $L_{01}\circ L_{12}$ does not lift to $L_{01}\times_{M_1} L_{12}$, or a figure eight bubble with $\lim_{s\to-\infty} w_1^\infty(s,-) \neq \lim_{s\to+\infty} w_1^\infty(s,-)$. However, this is not a sufficient condition, since a tree involving several sheet-switching bubbles at z_j could allow continuous extension of $\overline{v}_1$ to z_j.

The proof of Theorem 3.3.1 will take up the rest of this section. Our first goal will be to find a subsequence and blow-up points so that (2) and (3) hold together with the following bound on energy densities:

(0) The energy densities $|d\underline{v}^\nu|$ are uniformly bounded away from the bubbling points, that is for each compact subset $K \subset (-\rho,\rho)^2 \setminus \{z_1, \dots z_N\}$ we have

$$\sup_{\nu\in\mathbb{N}} \left\| d\underline{v}^\nu \right\|_{L^\infty(K\cap(-\rho,\rho)^2)} < \infty.$$

Then we will show that (1) follows from Theorem 3.2.9.

Suppose that we have already found a subsequence (for convenience again indexed by $\nu \in \mathbb{N}$) and some blow-up points $z_1, \dots, z_N \in (-\rho,\rho)^2$ such that (2) holds and we have established (3) at each such point. Now either (0) holds, too, or we can pass to a further subsequence and find another blow-up point $z_{N+1} = (s_{N+1}, t_{N+1}) = \lim_{\nu\to\infty}(s^\nu, t^\nu) \in (-\rho,\rho)^2 \setminus \{z_1, \dots z_N\}$ such that $\lim_{\nu\to\infty} |d\underline{v}^\nu(s^\nu,t^\nu)| = \infty$. We can apply the Hofer trick [McSa, Lemma 4.3.4][10] to vary the points (s^ν, t^ν) slightly (not changing their limit) and find $\epsilon^\nu \to 0$ such that we have

$$\sup_{(s,t)\in B_{\epsilon^\nu}(s^\nu,t^\nu)} |d\underline{v}^\nu(s,t)| \le 2|d\underline{v}^\nu(s^\nu,t^\nu)| =: 2R^\nu, \qquad R^\nu\epsilon^\nu \to \infty. \tag{3.17}$$

We will essentially rescale by R^ν around (s^ν, t^ν) to obtain different types of bubbles, depending on where the lines $\{t = \pm f^\nu(s)\}$ get mapped under the rescaling. We denote by $\tau_\pm^\nu := R^\nu(\pm f^\nu(s^\nu) - t^\nu)$ the t-coordinate of the preimage of the point $(s^\nu, \pm f^\nu(s^\nu))$ under the rescaling $t \mapsto t^\nu + t/R^\nu$. After passing to a subsequence, we may assume that $\tau_\pm^\nu$ converges to $\tau_\pm^\infty \in \mathbb{R} \cup \{\pm\infty\}$ with $\tau_-^\infty \le \tau_+^\infty$. Then exactly one of the following cases holds:

(S0) $\tau_-^\infty = \tau_+^\infty = \infty$

(S1) $\tau_-^\infty = -\infty$ and $\tau_+^\infty = \infty$

(S2) $\tau_-^\infty = \tau_+^\infty = -\infty$

(D01) $\tau_-^\infty \in \mathbb{R}$ and $\tau_+^\infty = \infty$

(D12) $\tau_-^\nu = -\infty$ and $\tau_+^\nu \in \mathbb{R}$

(E012) $\tau_\pm^\infty \in \mathbb{R}$ and $\tau_-^\infty < \tau_+^\infty$

(D02) $\tau_-^\infty = \tau_+^\infty \in \mathbb{R}$

Below, we will for each case specify the rescaled maps and establish their convergence to one of the bubble types in (3) as well as prove the energy concentration in (2). Thus in all cases we will have proven (2) and (3) for the new blow-up point z_{N+1}, and after adding this point we will either have (0) satisfied or be able to find another blow-up point. Since $\hbar > 0$ by Lemma 3.2.8, we will find at most $E/\hbar$ such blow-up points in this iteration before (0) holds.

Out of the seven blow-up scenarios just listed, the only case where our rescaling argument will be significantly different from standard rescaling arguments is (D02), in which we will need to appeal to the new analysis of Theorem 3.2.9. The rescaling argument in the cases (D01), (D12), (E012) is essentially the same as the standard process of "bubbling off a disk",

[10] Note that the Hofer trick applies directly to each function $f(x) = |d\underline{v}^\nu(x)|$ for $x = (s,t)$, although it is only upper semi-continuous. In the proof, continuity is used only to exclude $f(x_n) \to \infty$ for a convergent sequence $x_n \to x_\infty$. For a bounded upper semi-continuous function f, we still have $\limsup f(x_n) \le f(x_\infty) < \infty$, excluding this divergence.

since locally we can fold across the seam to obtain a pseudoholomorphic map to a product manifold.

Before we rescale to obtain bubbles, we record the key properties of the rescaled width function.

Lemma 3.3.3. *Given a sequence $(f^\nu)_{\nu\in\mathbb{N}}$ of real-analytic functions shrinking obediently to zero, shifts $s^\nu \to s^\infty$, and rescaling factors $\alpha^\nu \to \infty$, the rescaled width functions $\widetilde{f}^\nu(s) := \alpha^\nu f^\nu(s^\nu + s/\alpha^\nu)$ satisfy $\mathcal{C}^\infty_{\rm loc}(\mathbb{R})$ convergence*

$$\widetilde{f}^\nu - \widetilde{f}^\nu(0) \underset{\nu\to\infty}{\longrightarrow} 0, \qquad \widetilde{f}^\nu/\widetilde{f}^\nu(0) \underset{\nu\to\infty}{\longrightarrow} 1.$$

Moreover, let F^ν be the extension of f^ν from Definition 3.2.2, identify $(s,t)\in\mathbb{R}^2$ with $z = s+it \in \mathbb{C}$, and set

$$\begin{aligned} \phi^\nu(s,t) &:= \bigl(s^\nu + s/\alpha^\nu, 2f^\nu(s^\nu + s/\alpha^\nu)\, t\bigr) \\ \psi^\nu(z) &:= s^\nu + z/\alpha^\nu - iF^\nu\bigl(s^\nu + z/\alpha^\nu\bigr). \end{aligned}$$

Then for any $R>0$ and ν sufficiently large, the maps $(\psi^\nu)^{-1}\circ\phi^\nu$ are well defined on $B_R(0)$. In the special case $\alpha^\nu := (2f^\nu(s^\nu))^{-1}$, the maps $(\psi^\nu)^{-1}\circ\phi^\nu$ converge $\mathcal{C}^\infty_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2)$ to $(s,t)\mapsto(s,t+1/2)$.

Proof. The functions $\widetilde{f}^\nu(s) - \widetilde{f}^\nu(0)$ resp. $\widetilde{f}^\nu(s)/\widetilde{f}^\nu(0)$ are equal to 0 resp. 1 at $s=0$, so it suffices to show that $(\widetilde{f}^\nu(s) - \widetilde{f}^\nu(0))^{(k)}$ and $(\widetilde{f}^\nu(s)/\widetilde{f}^\nu(0))^{(k)}$ converge $\mathcal{C}^0_{\rm loc}$ to 0 for every $k\geq 1$. This convergence follows from the formulas

$$(\widetilde{f}^\nu - \widetilde{f}^\nu(0))^{(k)}(s) = (\alpha^\nu)^{1-k}(f^\nu)^{(k)}(s^\nu + \tfrac{s}{\alpha^\nu}), \quad (\widetilde{f}^\nu/\widetilde{f}^\nu(0))^{(k)}(s) = \frac{(\alpha^\nu)^{-k}(f^\nu)^{(k)}(s^\nu + \frac{s}{\alpha^\nu})}{f^\nu(s^\nu)},$$

the convergence $s^\nu \to s^\infty$ and $\alpha^\nu\to\infty$, and the obedient shrinking $f^\nu \Rightarrow 0$ in Definition 3.2.2.

The domain of ϕ^ν is $[-\alpha^\nu(s^\nu+\rho), -\alpha^\nu(s^\nu-\rho)]\times\mathbb{R}$ which contains $B_R(0)$ for sufficiently large ν since $s^\nu\to s^\infty\in(-\rho,\rho)$, $\alpha^\nu\to\infty$, and $f^\nu \xrightarrow{\mathcal{C}^\infty} 0$. The image concentrates at $(s^\nu,0)$, more precisely we have

$$\phi^\nu(B_R(0)) \subset B_{R\delta^\nu}(s^\nu,0), \qquad \delta^\nu := \max\bigl\{(\alpha^\nu)^{-1}, 2\|f^\nu\|_{\mathcal{C}^0}\bigr\} \to 0,$$

since $\bigl|\phi^\nu(s,t) - (s^\nu,0)\bigr|^2 \leq s^2/(\alpha^\nu)^2 + 4\|f^\nu\|^2_{\mathcal{C}^0}t^2 \leq \bigl(\delta^\nu|(s,t)|\bigr)^2$. Next, we claim that $B_{R\delta^\nu}(s^\nu,0)$ lies in the image of ψ^ν for sufficiently large ν. Indeed, given $y\in B_{R\delta^\nu}(s^\nu,0)$, we can solve

$$y = \psi^\nu(z) \tag{3.18}$$

iff there is a solution $z\in -\alpha^\nu s^\nu + [-\alpha^\nu\rho, \alpha^\nu\rho]^2$ of

$$z = \alpha^\nu\Bigl(y - s^\nu + iF^\nu\bigl(s^\nu + z/\alpha^\nu\bigr)\Bigr) =: H(z).$$

The existence of a such a solution follows from Banach's fixed point theorem applied to H. Indeed, H is a smooth map from $-\alpha^\nu s^\nu + [-\alpha^\nu\rho,\alpha^\nu\rho]^2$ to itself since $y\in B_{R\delta^\nu}(s^\nu,0)$ gives

$$\bigl|H(z) + \alpha^\nu s^\nu\bigr| = \bigl|\alpha^\nu(y + iF^\nu(s^\nu + \tfrac{z}{\alpha^\nu}))\bigr| \leq \alpha^\nu(|s^\nu| + R\delta^\nu + \|F^\nu\|_{\mathcal{C}^0}) < \alpha^\nu\rho$$

for ν sufficiently large so that $R\delta^\nu + \|F^\nu\|_{C^0} < \rho - |s^\nu|$. The latter holds for large ν since the left hand side converges to 0 while $\rho - |s^\nu| \to \rho - |s^\infty| > 0$. Furthermore, H is a contraction mapping once ν is large enough so that $\|F^\nu\|_{C^1} < 1$,

$$\big|H(z) - H(w)\big| = \alpha^\nu \big|F^\nu(s^\nu + z/\alpha^\nu) - F^\nu(s^\nu + w/\alpha^\nu)\big| \leq \|F^\nu\|_{C^1}|z-w|.$$

Therefore Banach's fixed point theorem guarantees a (unique) solution $z \in -\alpha^\nu s^\nu + [-\alpha^\nu\rho, \alpha^\nu\rho]^2$ of (3.18), which shows that for $\nu \gg 1$, the image of ψ^ν contains $\phi^\nu(B_R(0))$. To show that $(\psi^\nu)^{-1} \circ \phi^\nu$ is a well-defined element of $\mathcal{C}^\infty(B_R(0), \mathbb{R}^2)$, it remains to show that ψ^ν is injective and has a Jacobian with nonvanishing determinant.

Injectivity again holds once $\|F^\nu\|_{C^1} < 1$ since

$$\begin{aligned}\psi^\nu(z) = \psi^\nu(w) &\iff z - w = i\alpha^\nu\Big(F\big(s^\nu + \tfrac{z}{\alpha^\nu}\big) - F\big(s^\nu + \tfrac{w}{\alpha^\nu}\big)\Big)\\ &\implies |z-w| \leq \|F^\nu\|_{C^1}|z-w|.\end{aligned}$$

The Jacobian of ψ^ν is given by

$$\mathfrak{J}^\nu(s,t) := \mathrm{Jac}(\psi^\nu)(s+it) = (\alpha^\nu)^{-1}\begin{pmatrix} 1 + \partial_s \operatorname{im} F^\nu(s^\nu + \frac{s+it}{\alpha^\nu}) & \partial_t \operatorname{im} F^\nu(s^\nu + \frac{s+it}{\alpha^\nu}) \\ -\partial_s \operatorname{re} F^\nu(s^\nu + \frac{s+it}{\alpha^\nu}) & 1 - \partial_t \operatorname{re} F^\nu(s^\nu + \frac{s+it}{\alpha^\nu}) \end{pmatrix}, \tag{3.19}$$

which for $\nu \gg 1$ has nonvanishing determinant since $F^\nu \xrightarrow{\mathcal{C}^\infty} 0$. This proves that $(\psi^\nu)^{-1} \circ \phi^\nu$ is a well-defined element of $\mathcal{C}^\infty(B_R(0), \mathbb{R}^2)$ for $\nu \gg 1$.

We now restrict to the case $\alpha^\nu := (2f^\nu(s^\nu))^{-1}$. To establish the $\mathcal{C}^\infty_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$-convergence of $(\psi^\nu)^{-1} \circ \phi^\nu$ to the map $(s,t) \mapsto (s, t + \frac{1}{2})$, we begin by noting their equality at $(s,t) = (0, -\frac{1}{2})$,

$$\big((\psi^\nu)^{-1} \circ \phi^\nu\big)(0, -\tfrac{1}{2}) = (\psi^\nu)^{-1}(s^\nu, -f^\nu(s^\nu)) = (0,0).$$

It remains to show $\mathcal{C}^\infty_{\mathrm{loc}}$-convergence of the Jacobians $\mathrm{Jac}\big((\psi^\nu)^{-1} \circ \phi^\nu\big) \to \mathrm{Id}$. Using the inverse of (3.19) and abbreviating $Q^\nu(s,t) := s^\nu + \big((\psi^\nu)^{-1} \circ \phi^\nu\big)(s,t)/\alpha^\nu$ we have

$$\begin{aligned}\mathrm{Jac}\big((\psi^\nu)^{-1} \circ \phi^\nu\big)(s,t) &= \Big(\mathfrak{J}^\nu\big((\psi^\nu)^{-1}(\phi^\nu(s,t))\big)\Big)^{-1} \cdot \mathrm{Jac}(\phi^\nu)(s,t)\\ &= \frac{\begin{pmatrix} 1 - \partial_t \operatorname{re} F^\nu \circ Q^\nu & -\partial_t \operatorname{im} F^\nu \circ Q^\nu \\ \partial_s \operatorname{re} F^\nu \circ Q^\nu & 1 + \partial_s \operatorname{im} F^\nu \circ Q^\nu \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2(f^\nu)'(s^\nu + s/\alpha^\nu)t & 2\alpha^\nu f^\nu(s^\nu + s/\alpha^\nu) \end{pmatrix}}{(1 + \partial_s \operatorname{im} F^\nu \circ Q^\nu)(1 - \partial_t \operatorname{re} F^\nu \circ Q^\nu) + \partial_t \operatorname{im} F^\nu \circ Q^\nu \partial_s \operatorname{re} F^\nu \circ Q^\nu}.\end{aligned} \tag{3.20}$$

The $\mathcal{C}^\infty$-convergence $F^\nu \to 0$ implies that the first matrix divided by the denominator converges $\mathcal{C}^0_{\mathrm{loc}}$ to the identity. In fact, this is $\mathcal{C}^k_{\mathrm{loc}}$-convergence if the derivatives of Q^ν up to order k are uniformly bounded on compact sets. In the second matrix we have $2(f^\nu)'(s^\nu + s/\alpha^\nu)t \to 0$ in $\mathcal{C}^\infty_{\mathrm{loc}}$ by the $\mathcal{C}^\infty$-convergence $f^\nu \to 0$ and $(\alpha^\nu)^{-1} \to 0$, and the bottom right entry $\frac{f^\nu(s^\nu + 2f^\nu(s^\nu)s)}{f^\nu(s^\nu)} = \frac{\tilde{f}^\nu(s)}{\tilde{f}^\nu(0)} \to 1$ converges already in $\mathcal{C}^\infty_{\mathrm{loc}}$ by the first statement of the current lemma.

This proves $\mathcal{C}^0_{\mathrm{loc}}$ convergence of the Jacobians and thus $\mathcal{C}^1_{\mathrm{loc}}$-convergence of the maps $(\psi^\nu)^{-1} \circ \phi^\nu$. Since Q^ν is given in terms of these maps and $(\alpha^\nu)^{-1} \to 0$, we conclude that its derivatives are uniformly bounded on compact sets, thus the convergence of the Jacobians is in $\mathcal{C}^1_{\text{loc}}$, which implies $\mathcal{C}^2_{\text{loc}}$-convergence of the maps $(\psi^\nu)^{-1}\circ\phi^\nu$. Iterating this argument proves the claimed $\mathcal{C}^\infty_{\text{loc}}$ convergence. $\square$

Continuing with the proof of Theorem 3.3.1, the nontrivial bubbles claimed in (3) are now obtained as follows:

(S1): We will obtain a **sphere bubble in $\mathbf{M_1}$** by rescaling

$$w_1^\nu(s,t) := v_1^\nu(s^\nu + \tfrac{s}{R^\nu}, t^\nu + \tfrac{t}{R^\nu})$$

to define maps $w_1^\nu\colon U_1^\nu \to M_1$, with

$$U_1^\nu := \big\{(s,t) \mid -R^\nu(\rho+s^\nu) \le s \le R^\nu(\rho - s^\nu),\ -\widetilde{f}^\nu(s) - R^\nu t^\nu \le t \le \widetilde{f}^\nu(s) - R^\nu t^\nu\big\}.$$

The map w_1^ν is pseudoholomorphic with respect to $\widetilde{J}_1^\nu(t) := J_1^\nu(s^\nu + s/R^\nu, t^\nu + t/R^\nu)$; due to the convergence $R^\nu\to\infty$, these almost complex structures converge in $\mathcal{C}^\infty_{\text{loc}}(\mathbb{R}^2)$ as $\nu\to\infty$ to the constant almost complex structure $\widetilde{J}_1^\infty := J_1^\infty(s^\infty,t^\infty)$. By construction, the maps w_1^ν satisfy both upper and lower gradient bounds:

$$\begin{aligned} |\mathrm{d}w_1^\nu(0)| &= \tfrac{1}{R^\nu}|\mathrm{d}v_1^\nu(s^\nu,t^\nu)| = \tfrac{1}{R^\nu}|\mathrm{d}\underline{v}(s^\nu,t^\nu)| = 1, \\ \sup_{(s,t)\in B_{R^\nu\epsilon^\nu}(0)} |\mathrm{d}w_1^\nu(s,t)| &\le \sup_{(s,t)\in B_{\epsilon^\nu}(s^\nu,t^\nu)} \tfrac{1}{R^\nu}|\mathrm{d}\underline{v}^\nu(s,t)| \le 2, \end{aligned} \tag{3.21}$$

where the second equality in the top line follows for large ν from the assumption $\tau_\pm^\nu\to\pm\infty$. The containment $s^\nu\to s^\infty\in(-\rho,\rho)$ implies that the left resp. right bounds $R^\nu(\mp\rho - s^\nu)$ of U_1^ν have limits $-\infty$ resp. ∞; furthermore, the assumption $\tau_\pm^\infty = \pm\infty$ and Lemma 3.3.3 implies that the lower resp. upper bounds $-\widetilde{f}^\nu(s) - R^\nu t^\nu$ resp. $\widetilde{f}^\nu(s) - R^\nu t^\nu$ of U_1^ν converge $\mathcal{C}^\infty_{\text{loc}}$ to $-\infty$ resp. ∞. Hence the maps w_1^ν are defined with uniformly bounded differential on balls centered at 0 of radii tending to infinity. Standard compactness for pseudoholomorphic maps (e.g. [McSa, Appendix B][11]) implies that a subsequence, still denoted by (w_1^ν), converges $\mathcal{C}^\infty_{\text{loc}}$ to a $\widetilde{J}_1^\infty$-holomorphic map w_1^∞ defined on $\mathbb{R}^2$. Its energy is bounded by E, so after removing the singularity (using [McSa, Theorem 4.1.2(i)]) we obtain a $\widetilde{J}_1^\infty$-holomorphic sphere $\overline{w}_1^\infty\colon S^2\to M_1$, which is nonconstant by (3.21). Rescaling invariance of the energy and $\mathcal{C}^\infty_{\text{loc}}$-convergence imply energy concentration:

$$\begin{aligned} \liminf_{\nu\to\infty}\int_{B_{\epsilon^\nu}(s^\nu,t^\nu)} \tfrac12|\mathrm{d}\underline{v}^\nu|^2 \ge \liminf_{\nu\to\infty}\int_{B_{\epsilon^\nu}(s^\nu,t^\nu)} {v_1^\nu}^*\omega_1 &\ge \liminf_{\nu\to\infty}\int_{B_{R^\nu\epsilon^\nu}(0)\cap U_1^\nu} {w_1^\nu}^*\omega_1 \\ &\ge \int_{\mathbb{R}^2} {w_1^\infty}^*\omega_1 \\ &\ge \hbar_{S^2} > 0. \end{aligned}$$

(S0,S2): In complete analogy to (S1), rescaling by $w_\ell^\nu(s,t) := v_\ell^\nu(s^\nu + s/R^\nu, t^\nu + t/R^\nu)$ yields a nonconstant pseudoholomorphic **sphere in $\mathbf{M}_\ell$** and with energy concentration of at least $\hbar_{S^2}$.

[11] If noncompact symplectic manifolds are involved, then one needs to establish $\mathcal{C}^0$-bounds on the maps before "standard Gromov compactness" can be quoted.

(D01): We will obtain a **disk bubble in $\mathbf{M_0^-} \times \mathbf{M_1}$ with boundary on $\mathbf{L_{01}}$** by rescaling

$$w_\ell^\nu(z) := v_\ell^\nu\Big(s^\nu + \tfrac{s}{R^\nu}, -f^\nu(s^\nu + \tfrac{s}{R^\nu}) + \tfrac{t}{R^\nu}\Big)$$

to define maps

$$\begin{aligned} w_0^\nu &: \big\{(s,t) \mid -R^\nu(\rho + s^\nu) \le R^\nu(\rho - s^\nu),\ R^\nu\big(-\rho + f^\nu(s^\nu + \tfrac{s}{R^\nu})\big) \le t \le 0\big\} \longrightarrow M_0, \\ w_1^\nu &: \big\{(s,t) \mid -R^\nu(\rho + s^\nu) \le R^\nu(\rho - s^\nu),\ 0 \le t \le 2R^\nu f^\nu(s^\nu + \tfrac{s}{R^\nu})\big\} \longrightarrow M_1. \end{aligned}$$

In the case of straight seams $t = \pm f^\nu(s) = \pm\delta^\nu$ the rescaled maps easily pair to maps $w_{01}^\nu : (s,t) \mapsto \big(w_0^\nu(-s,t), w_1^\nu(s,t)\big) \in M_0^- \times M_1$ defined on increasing domains $B_{r^\nu}(0) \cap \mathbb{H}$ with $r^\nu \to \infty$ in half space, with boundary values in L_{01}, which by standard arguments converge and extend to a pseudholomorphic disk. The squiggly seams require an easier version of the arguments in (E012) to establish convergence $w_0^\nu \to w_0^\infty$, $w_1^\nu \to w_1^\infty$ in $\mathcal{C}^\infty_{\rm loc}(-\mathbb{H})$ resp. $\mathcal{C}^\infty_{\rm loc}(\mathbb{H})$ to nonconstant $J_\ell^\infty(s^\infty, 0)$-holomorphic maps satisfying the Lagrangian seam condition

$$(w_0^\infty(s,0), w_1^\infty(s,0)) \in L_{01} \quad \forall s \in \mathbb{R}.$$

Then $w_{01}^\infty(s,t) := (w_0^\infty(s,-t), w_1^\infty(s,t)) \colon \mathbb{H} \to M_0^- \times M_1$ is a nonconstant $\widetilde{J}_{01}^\infty := (-J_0^\infty(s^\infty, 0)) \times J_1^\infty(s^\infty, 0)$-holomorphic map with $w_{01}^\infty(\partial\mathbb{H}) \subset L_{01}$. Its energy is bounded by E, so after removing the singularity (using e.g. [McSa, Theorem 4.1.2(ii)]) we obtain a nonconstant $\widetilde{J}_{01}^\infty$-holomorphic disk $\overline{w}_{01}^\infty \colon D^2 \to M_0^- \times M_1$. Energy quantization in the case of straight seams is given by

$$\begin{aligned} \liminf_{\nu\to\infty} \int_{B_{\epsilon^\nu}(s^\nu,t^\nu)} \tfrac12 |\mathrm{d}\underline{v}^\nu|^2 &\ge \liminf_{\nu\to\infty} \sum_{\ell\in\{0,1\}} \int_{B_{\epsilon^\nu}(s^\nu,t^\nu)} v_\ell^{\nu\,*}\omega_\ell \\ &\ge \liminf_{\nu\to\infty} \int_{B_{r^\nu}(0)\cap\mathbb{H}} w_{01}^{\nu\,*}\big((-\omega_0) \oplus \omega_1\big) \\ &\ge \int w_{01}^{\infty\,*}\big((-\omega_0) \oplus \omega_1\big) \ge \hbar_{D^2}, \end{aligned}$$

and in the general case just requires more refined choices of domains as in (E012).

(D12): In complete analogy to (D01), rescaling yields a nonconstant pseudoholomorphic **disk in $\mathbf{M_1^-} \times \mathbf{M_2}$ with boundary on $\mathbf{L_{12}}$** and energy concentration of at least $\hbar_{D^2}$.

(E012): We will obtain a **figure eight bubble between $\mathbf{L_{01}}$ and $\mathbf{L_{12}}$** by a rescaling that in the case of straight middle strips of width $2f^\nu \equiv \delta^\nu \to 0$ amounts to $w_\ell^\nu(s,t) = v_\ell^\nu\big(s^\nu + \delta^\nu s, \delta^\nu t\big)$ for $\ell = 0,1,2$. In the general case of squiggly strips, we straighten out the strip by using an s-dependent rescaling factor in the t variable:

$$w_\ell^\nu(s,t) := v_\ell^\nu\big(\phi^\nu(s,t)\big), \qquad \phi^\nu(s,t) := \Big(s^\nu + 2f^\nu(s^\nu)s, 2f^\nu\big(s^\nu + 2f^\nu(s^\nu)s\big)\, t\Big). \tag{3.22}$$

Note that ϕ^ν is a diffeomorphism between open subsets of $\mathbb{R}^2$ since $f^\nu > 0$, and that it pulls back the seams $\mathcal{S}_\pm = \big\{(x,y) \in \mathbb{R}^2 \,|\, y = \pm f^\nu(x)\big\}$ to straight seams $(\phi^\nu)^{-1}(\mathcal{S}_\pm) = \big\{(s,t) \in \mathbb{R}^2 \,|\, t = \pm\frac12\big\}$ as in Definition 3.2.5 of the figure eight bubble. Moreover, the rescaled quilt maps $(w_\ell^\nu)_{\ell=0,1,2}$ have the total domain

$$(\phi^\nu)^{-1}\big((-\rho,\rho)^2\big) = \Big\{(s,t)\in\mathbb{R}^2 \,\Big|\, \tfrac{-\rho-s^\nu}{2f^\nu(s^\nu)} < s < \tfrac{\rho-s^\nu}{2f^\nu(s^\nu)},\ |t| < \tfrac{\rho}{2f^\nu(s^\nu+2f^\nu(s^\nu)s)}\Big\}$$

Since $s^\nu \to s^\infty \in (-\rho,\rho)$ and $f^\nu \xrightarrow{\mathcal{C}^\infty} 0$, we can find a sequence of radii $r^\nu \to \infty$ so that these domains contain the balls $B_{r^\nu}(0)$. In the case of straight seams the maximal radii are $r^\nu = (\rho - |s^\nu|)/\delta^\nu$, but we may also choose smaller radii $r^\nu \to \infty$ so that in addition $r^\nu\delta^\nu \to 0$. In general we choose $r^\nu \to \infty$ so that $B_{r^\nu}(0) \subset (\phi^\nu)^{-1}\big((-\rho,\rho)^2\big)$ and

$$r^\nu \max_{s\in[-\rho,\rho]} \big(f^\nu(s) + (f^\nu)'(s)\big) \underset{\nu\to\infty}{\longrightarrow} 0. \tag{3.23}$$

Finally, we wish to choose $r^\nu \to \infty$ such that in addition the inclusion $\phi^\nu\big(B_{r^\nu}(0)\big) \subset B_{\epsilon^\nu}(s^\nu,t^\nu)$ ensures that the gradient bounds (3.17) transfer to the rescaled maps. For that purpose first note that for sufficiently large $\nu\in\mathbb{N}$ from (3.23) we also obtain the estimate

$$\max_{s\in[-r^\nu,r^\nu]} 2f^\nu(s^\nu + 2f^\nu(s^\nu)s) \;\le\; 3f^\nu(s^\nu). \tag{3.24}$$

Indeed, for $s\in[-r^\nu,r^\nu]$ and ν sufficiently large such that $r^\nu\|(f^\nu)'\|_{\mathcal{C}^0([-\rho,\rho])} \le \frac14$ we have

$$\begin{aligned} f^\nu(s^\nu + 2f^\nu(s^\nu)s) &\le f^\nu(s^\nu) + \int_0^s 2f^\nu(s^\nu)(f^\nu)'(s^\nu + 2f^\nu(s^\nu)s)\,ds \\ &\le f^\nu(s^\nu)\big(1 + 2r^\nu\|(f^\nu)'\|_{\mathcal{C}^0([-\rho,\rho])}\big) \;\le\; \tfrac32 f^\nu(s^\nu). \end{aligned}$$

Next, for $(s,t)\in B_{r^\nu}(0)$ we obtain

$$\begin{aligned} \big|(s^\nu + 2f^\nu(s^\nu)s, 2f^\nu(s^\nu+2f^\nu(s^\nu)s)t) - (s^\nu,t^\nu)\big| &\le |(2f^\nu(s^\nu)s, 2f^\nu(s^\nu+2f^\nu(s^\nu)s)t)| \\ &\qquad + |(0,t^\nu)| \\ &\le 3r^\nu f^\nu(s^\nu) + |t^\nu| \\ &= \frac{3r^\nu(\tau_+^\nu - \tau_-^\nu) + |\tau_+^\nu + \tau_-^\nu|}{2R^\nu\epsilon^\nu}\,\epsilon^\nu \end{aligned}$$

from (3.24) and the identities

$$f^\nu(s^\nu) = \frac{\tau_+^\nu - \tau_-^\nu}{2R^\nu}, \qquad t^\nu = -\frac{\tau_+^\nu + \tau_-^\nu}{2R^\nu}. \tag{3.25}$$

Thus to obtain the inclusion $\phi^\nu\big(B_{r^\nu}(0)\big) \subset B_{\epsilon^\nu}(s^\nu,t^\nu)$ for large ν it suffices to replace the above r^ν by a possibly smaller sequence $r^\nu \to \infty$ so that $3r^\nu(\tau_+^\nu - \tau_-^\nu) + |\tau_+^\nu + \tau_-^\nu| < 2R^\nu\epsilon^\nu$ for large ν. This is possible since $R^\nu\epsilon^\nu \to \infty$ and $\tau_\pm^\nu \to \tau_\pm^\infty \in \mathbb{R}$. So from now on, after dropping finitely many terms from the sequence, we are working with a sequence of rescaled quilt maps (3.22) together with a sequence $r^\nu \to \infty$ so that we have the inequalities (3.23), (3.24), and the inclusions

$$\phi^\nu\big(B_{r^\nu}(0)\big) \subset B_{\epsilon^\nu}(s^\nu,t^\nu) \cap (-\rho,\rho)^2. \tag{3.26}$$

Thus, restricting the maps from (3.22) to the balls $B_{r^\nu}(0)$ of increasing radius amounts to considering the quilt map $w^\nu_\ell \colon W^\nu_\ell \to M_\ell$ with the domains

$$\begin{aligned} W^\nu_0 &:= B_{r^\nu}(0) \cap \big(\mathbb{R} \times (-\infty, -\tfrac12]\big), \\ W^\nu_1 &:= B_{r^\nu}(0) \cap \big(\mathbb{R} \times [-\tfrac12, \tfrac12]\big), \\ W^\nu_2 &:= B_{r^\nu}(0) \cap \big(\mathbb{R} \times [\tfrac12, \infty)\big). \end{aligned}$$

These maps are $(\widetilde{J}^\nu_\ell, j^\nu)$-holomorphic, where $\widetilde{J}^\nu_\ell = J^\nu_\ell \circ \phi^\nu$ are the almost complex structures on M_ℓ with appropriately rescaled domain dependence, and j^ν is the complex structure

$$\begin{aligned} j^\nu(s,t) &:= \big(\mathrm{d}\phi^\nu(s,t)\big)^{-1} \circ j_0 \circ \mathrm{d}\phi^\nu(s,t) \\ &= \begin{pmatrix} 2f^\nu(s^\nu) & 0 \\ 4f^\nu(s^\nu)(f^\nu)'(s^\nu + 2f^\nu(s^\nu)s)\,t & 2f^\nu(s^\nu + 2f^\nu(s^\nu)s) \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times \\ &\qquad\qquad \times \begin{pmatrix} 2f^\nu(s^\nu) & 0 \\ 4f^\nu(s^\nu)(f^\nu)'(s^\nu + 2f^\nu(s^\nu)s)\,t & 2f^\nu(s^\nu + 2f^\nu(s^\nu)s) \end{pmatrix} \\ &= \begin{pmatrix} (2f^\nu(s^\nu))^{-1} & 0 \\ -\frac{(f^\nu)'(\cdots)\,t}{f^\nu(s^\nu+\cdots)} & (2f^\nu(s^\nu + \cdots))^{-1} \end{pmatrix} \times \\ &\qquad\qquad \times \begin{pmatrix} -4f^\nu(s^\nu)(f^\nu)'(s^\nu + \cdots)\,t & -2f^\nu(s^\nu + 2f^\nu(s^\nu)s) \\ 2f^\nu(s^\nu) & 0 \end{pmatrix} \\ &= \begin{pmatrix} -2t(f^\nu)'(s^\nu + 2f^\nu(s^\nu)s) & -\frac{f(s^\nu + 2f^\nu(s^\nu)s)}{f^\nu(s^\nu)} \\ \frac{f^\nu(s^\nu)\left(4t^2(f^\nu)'(s^\nu+2f^\nu(s^\nu)s)^2+1\right)}{f^\nu(s^\nu+2f^\nu(s^\nu)s)} & 2t(f^\nu)'(s^\nu + 2f^\nu(s^\nu)s) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{t(\widetilde{f}^\nu)'(s)}{\widetilde{f}^\nu(0)} & -\frac{\widetilde{f}^\nu(s)}{\widetilde{f}^\nu(0)} \\ \frac{t^2(\widetilde{f}^\nu)'(s)^2 + \widetilde{f}^\nu(0)^2}{\widetilde{f}^\nu(0)\widetilde{f}^\nu(s)} & \frac{t(\widetilde{f}^\nu)'(s)}{\widetilde{f}^\nu(0)} \end{pmatrix}, \end{aligned}$$

where we abbreviate $\widetilde{f}^\nu(s) := f^\nu(s^\nu + 2f^\nu(s^\nu)s)/2f^\nu(s^\nu)$. Note that Lemma 3.3.3 with $\alpha^\nu := (2f^\nu(s^\nu))^{-1}$ implies $j^\nu \to i$ in $\mathcal{C}^\infty_{\mathrm{loc}}$, and the almost complex structures also converge $\widetilde{J}^\nu_\ell \to J^\infty_\ell(s^\infty, 0)$ in $\mathcal{C}^\infty_{\mathrm{loc}}$ since $\phi^\nu(s,t) \to 0$ for any fixed (s,t). Moreover, the maps w^ν_ℓ satisfy the Lagrangian seam conditions

$$(w^\nu_0(s, -\tfrac12), w^\nu_1(s, -\tfrac12)) \in L_{01}, \quad (w^\nu_1(s, \tfrac12), w^\nu_2(s, \tfrac12)) \in L_{12} \qquad \forall\, s \in (-r^\nu, r^\nu).$$

The gradient blowup at (s^ν, t^ν) in (3.17) translates into lower bounds on the gradient $|\mathrm{d}w^\nu| := \big(|\mathrm{d}w^\nu_0| + |\mathrm{d}w^\nu_1| + |\mathrm{d}w^\nu_2|\big)^{1/2}$ at $t = \widetilde{t}^\nu := \frac{t^\nu}{2f^\nu(s^\nu)} = \frac{-\tau^\nu_+ - \tau^\nu_-}{2(\tau^\nu_+ - \tau^\nu_-)} \to \frac{-\tau^\infty_+ - \tau^\infty_-}{2(\tau^\infty_+ - \tau^\infty_-)} \in \mathbb{R}$, since $\phi^\nu(0, \frac{t^\nu}{2f^\nu(s^\nu)}) = (s^\nu, t^\nu)$ and hence

$$\begin{aligned} \big|\mathrm{d}\underline{w}^\nu(0, \widetilde{t}^\nu)\big|^2 &= \sum_\ell \Big(\big|2f^\nu(s^\nu)\partial_s v^\nu_\ell(s^\nu, t^\nu) + 4\widetilde{t}^\nu f^\nu(s^\nu)(f^\nu)'(s^\nu)\partial_t v^\nu_\ell(s^\nu, t^\nu)\big|^2 \\ &\qquad\qquad + \big|2f^\nu(s^\nu)\partial_t v^\nu_\ell(s^\nu, t^\nu)\big|^2 \Big) \\ &\geq 3\big|\mathrm{d}\underline{v}^\nu(s^\nu, t^\nu)\big|^2, \quad \nu \gg 0. \end{aligned}$$

Now by $t^\nu \to t^\infty$, the obedient convergence $f^\nu \Rightarrow 0$ in Definition 3.2.2, and (3.25) we obtain a nonzero lower bound for sufficiently large ν,

$$\big|\mathrm{d}\underline{w}^\nu(0,\widetilde{t}^\nu)\big|^2 \;\geq\; 2f^\nu(s^\nu)^2|\mathrm{d}\underline{v}^\nu(s^\nu,t^\nu)|^2 \;=\; \tfrac{1}{2}(\tau_+^\nu-\tau_-^\nu)^2 \;\geq\; \tfrac{1}{4}(\tau_+^\infty-\tau_-^\infty)^2>0. \qquad (3.27)$$

Next we use (3.23)–(3.26) to transfer the upper bound in (3.17) to the gradient of the rescaled maps for sufficiently large ν,

$$\begin{aligned}
&\sup_{(s,t)\in B_{r^\nu}(0)} |\mathrm{d}\underline{w}^\nu(s,t)|^2 \\
&= \sup_{(s,t)\in B_{r^\nu}(0)} \sum_\ell \begin{aligned}[t] &\big|2f^\nu(s^\nu)\partial_s v_\ell^\nu(\phi^\nu(s,t)) + 4tf^\nu(s^\nu)(f^\nu)'(s^\nu+2f^\nu(s^\nu)s)\partial_t v_\ell^\nu(\phi^\nu(s,t))\big|^2 \\ &\qquad + \big|2f^\nu(s^\nu+2f^\nu(s^\nu)s)\partial_t v_\ell^\nu(\phi^\nu(s,t))\big|^2\end{aligned} \\
&\leq \Big(\big(2f^\nu(s^\nu)+4r^\nu f^\nu(s^\nu)\max_{s\in[-\rho,\rho]}|(f^\nu)'(s)|\big)^2 + \big(3f^\nu(s^\nu)\big)^2\Big) \sup_{(x,y)\in B_{\epsilon^\nu}(s^\nu,t^\nu)} |\mathrm{d}\underline{v}(x,y)|^2 \\
&\leq 18f^\nu(s^\nu)^2 \sup_{(x,y)\in B_{\epsilon^\nu}(s^\nu,t^\nu)} |\mathrm{d}\underline{v}(x,y)|^2 \;\leq\; 18(\tau_+^\infty-\tau_-^\infty)^2. \qquad (3.28)
\end{aligned}$$

Using these gradient bounds (and the compact boundary conditions in the case of noncompact symplectic manifolds), standard Gromov compactness asserts that after passing to a subsequence, w_0^ν resp. w_1^ν resp. w_2^ν converge $\mathcal{C}^\infty_{\mathrm{loc}}$ on the interior of $\mathbb{R}\times(-\infty,-1/2]$ resp. $\mathbb{R}\times[-1/2,1/2]$ resp. $\mathbb{R}\times[1/2,\infty)$.

To obtain convergence up to the seams $\mathbb{R}\times\{\pm1/2\}$, we will first prove convergence of somewhat differently rescaled maps. More precisely, to prove convergence of w_0^ν, w_1^ν near the L_{01}-seam $\mathbb{R}\times\{-1/2\}$ we consider the maps

$$u_0^\nu\colon\; U_0^\nu := (-r^\nu,r^\nu)\times(-1/2,0] \;\to\; M_0, \qquad u_1^\nu\colon\; U_1^\nu := (-r^\nu,r^\nu)\times[0,1/2) \;\to\; M_1$$

given by rescaling $u_\ell^\nu(s,t) := v_\ell^\nu\big(\psi^\nu(s+it)\big)$ with the holomorphic map

$$\psi^\nu(z) := s^\nu + 2f^\nu(s^\nu)z - iF^\nu\big(s^\nu+2f^\nu(s^\nu)z\big),$$

where we identify $(s,t)\in\mathbb{R}^2$ with $z=s+it\in\mathbb{C}$, and F^ν is the extension of f^ν from Definition 3.2.2. To see that ψ^ν is well-defined on $U_0^\nu\cup U_1^\nu$ for sufficiently large ν, despite f^ν resp. F^ν only being defined on $[-\rho,\rho]$ resp. $[-\rho,\rho]^2$, note that $s^\nu\to s^\infty\in(-\rho,\rho)$ and $r^\nu f^\nu(s^\nu)\to 0$ by (3.23). To ensure that u_ℓ^ν is well-defined for large ν and $\ell=0,1$ we moreover need to verify that $\psi^\nu(U_\ell^\nu)$ lies in the domain of v_ℓ^ν. Indeed, firstly we have $\psi^\nu(U_0^\nu\cup U_1^\nu)\subset(-\rho,\rho)^2$ for large ν by $s^\nu\to s^\infty\in(-\rho,\rho)$, (3.23), and $F^\nu \xrightarrow{\mathcal{C}^\infty} 0$. Secondly, the bounds required by the seams are

$$\begin{aligned}
\operatorname{im}\psi^\nu(s+it) &\leq -f^\nu(\operatorname{re}\psi^\nu(s+it)) \quad && \forall\, s+it\in U_0^\nu, \qquad (3.29)\\
\big|\operatorname{im}\psi^\nu(s+it)\big| &\leq f^\nu(\operatorname{re}\psi^\nu(s+it)) && \forall\, s+it\in U_1^\nu \qquad (3.30)
\end{aligned}$$

for large ν. For that purpose we rewrite

$$\begin{aligned}
&\operatorname{im}\psi^\nu(s+it) \pm f^\nu(\operatorname{re}\psi^\nu(s+it)) \\
&= 2tf^\nu(s^\nu) - \operatorname{re}F^\nu\big(s^\nu+2f^\nu(s^\nu)(s+it)\big) \\
&\qquad\qquad \pm f^\nu\Big(s^\nu+2f^\nu(s^\nu)s + \operatorname{im}F^\nu\big(s^\nu+2f^\nu(s^\nu)(s+it)\big)\Big) \\
&= (2t-1\pm1)f^\nu\big(s^\nu+2f^\nu(s^\nu)s\big) + E_1^\nu(s,t) - \operatorname{re}E_2^\nu(s,t) \pm E_3^\nu(s,t).
\end{aligned}$$

with

$$\begin{aligned}
E_1^\nu(s,t) &= 2t\big(f^\nu(s^\nu) - f^\nu\big(s^\nu + 2f^\nu(s^\nu)s\big)\big),\\
E_2^\nu(s,t) &= F^\nu\big(s^\nu + 2f^\nu(s^\nu)(s+it)\big) - F^\nu\big(s^\nu + 2f^\nu(s^\nu)s\big),\\
E_3^\nu(s,t) &= f^\nu\Big(s^\nu + 2f^\nu(s^\nu)s + \operatorname{im} F^\nu\big(s^\nu + 2f^\nu(s^\nu)(s+it)\big)\Big) - f^\nu\big(s^\nu + 2f^\nu(s^\nu)s\big).
\end{aligned}$$

We can bound $E_1^\nu, E_2^\nu, E_3^\nu$ for large ν and $(s,t) \in U_0^\nu \cup U_1^\nu \subset B_{2r^\nu}(0)$, using the obedient convergence $f^\nu \Rightarrow 0$ from Definition 3.2.2,

$$|E_1^\nu(s,t)| \le 4t\|(f^\nu)'\|_{C^0} f^\nu(s^\nu)s \le 8tr^\nu\|(f^\nu)'\|_{C^0} C_0 f^\nu\big(s^\nu + 2f^\nu(s^\nu)s\big), \tag{3.31}$$

$$|E_2^\nu(s,t)| \le \|DF^\nu\|_{C^0} 2tf^\nu(s^\nu) \le 2t\|DF^\nu\|_{C^0} C_0 f^\nu\big(s^\nu + 2f^\nu(s^\nu)s\big), \tag{3.32}$$

$$\begin{aligned}
|E_3^\nu(s,t)| &\le \|(f^\nu)'\|_{C^0}\big|\operatorname{im} F^\nu(s^\nu + 2f^\nu(s^\nu)(s+it))\big| \\
&= \|(f^\nu)'\|_{C^0}\Big|\operatorname{im}\big(F^\nu(s^\nu + 2f^\nu(s^\nu)(s+it)) - F^\nu(s^\nu + 2f^\nu(s^\nu)s)\big)\Big| \\
&\le \|(f^\nu)'\|_{C^0}\|DF^\nu\|_{C^0} 2tf^\nu(s^\nu) \\
&\le 2t\|(f^\nu)'\|_{C^0}\|DF^\nu\|_{C^0} C_0 f^\nu\big(s^\nu + 2f^\nu(s^\nu)s\big).
\end{aligned} \tag{3.33}$$

Then by (3.23) and $F^\nu \xrightarrow{C^\infty} 0$ we obtain for sufficiently large ν

$$\big|\operatorname{im}\psi^\nu(s+it) \pm f^\nu(\operatorname{re}\psi^\nu(s+it)) - (2t-1\pm 1)f^\nu\big(s^\nu + 2f^\nu(s^\nu)s\big)\big| \le tf^\nu\big(s^\nu + 2f^\nu(s^\nu)s\big).$$

To check (3.29) from this, recall that $t \in (-1/2, 0]$ on U_0^ν so that

$$\operatorname{im}\psi^\nu(s+it) + f^\nu(\operatorname{re}\psi^\nu(s+it)) \le tf^\nu\big(s^\nu + 2f^\nu(s^\nu)s\big) \le 0.$$

Similarly, on U_1^ν we have $t \in [0, 1/2)$ so that (3.30) follows from

$$f^\nu(\operatorname{re}\psi^\nu(s+it)) \pm \operatorname{im}\psi^\nu(s+it) \ge \big(1 - t \pm (2t-1)\big)f^\nu\big(s^\nu + 2f^\nu(s^\nu)s\big) \ge 0$$

since $f^\nu > 0$, $1 - t - 2t + 1 = 2 - 3t > 0$ and $1 - t + 2t - 1 = t \ge 0$.

Now that u_0^ν, u_1^ν are well-defined, note that the advantage of this rescaling is that the resulting maps are pseudoholomorphic with respect to the standard complex structure i on their domains (viewed as subsets of $\mathbb{C}$). On the other hand it straightens out only one seam,

$$\begin{aligned}
\psi^\nu\big((-r^\nu, r^\nu) \times \{0\}\big) &= \big\{\big(s^\nu + 2f^\nu(s^\nu)s, -f^\nu\big(s^\nu + 2f^\nu(s^\nu)s\big)\big) \,\big|\, s \in (-r^\nu, r^\nu)\big\} \\
&\subset \big\{(x,y) \in \mathbb{R}^2 \,|\, y = -f^\nu(x)\big\} \;=\; \mathcal{S}_-,
\end{aligned}$$

so that we obtain the Lagrangian seam condition

$$(u_0^\nu(s,0), u_1^\nu(s,0)) \in L_{01} \qquad \forall\, s \in (-r^\nu, r^\nu),$$

but the L_{12}-condition would hold on the curved seam $(\psi^\nu)^{-1}(\mathcal{S}_+)$. However, we use this rescaling only to prove convergence near the L_{01}-seam, and to prove convergence of w_1^ν, w_2^ν near the L_{12}-seam would use the rescaling $z \mapsto s^\nu + 2f^\nu(s^\nu)z + iF^\nu\big(s^\nu + 2f^\nu(s^\nu)z\big)$.

More precisely, we will below prove $\mathcal{C}^\infty_{\mathrm{loc}}$-convergence of u^ν_ℓ near $\mathbb{R}\times\{0\}$ since this yields control of the maps of interest w^ν_ℓ for $\ell=0,1$ near the seam $\mathbb{R}\times\{-\frac{1}{2}\}$. Indeed, $w^\nu_\ell = u^\nu_\ell\circ((\psi^\nu)^{-1}\circ\phi^\nu)$ is obtained from u^ν_ℓ by composition with the local diffeomorphisms $(\psi^\nu)^{-1}\circ\phi^\nu$ which by Lemma 3.3.3 converge $\mathcal{C}^\infty_{\mathrm{loc}}$ to a shift map. On the other hand, to establish convergence of the u^ν_ℓ, we can start from local uniform gradient bounds given by (3.28) via the reparametrization with $(\psi^\nu)^{-1}\circ\phi^\nu$. Further, we can work with the "folded" maps $u^\nu_{01}\colon(-r^\nu,r^\nu)\times[0,1/2)\to M_0^-\times M_1$ given by $u^\nu_{01}(s,t):=(u^\nu_0(s,-t),u^\nu_1(s,t))$. These satisfy the Lagrangian boundary condition $u^\nu_{01}(s,0)\in L_{01}$ for $s\in(-r^\nu,r^\nu)$ and are pseudoholomorphic with respect to $K^\nu_{01}(s,t):=\bigl(-J^\nu_0\bigl(\psi^\nu(s-it)\bigr)\bigr)\times J^\nu_1\bigl(\psi^\nu(s+it)\bigr)$, which converges to $K^\infty_{01}:=\bigl(-J^\infty_0(s^\infty,0)\bigr)\times J^\infty_1(s^\infty,0)$ in $\mathcal{C}^\infty_{\mathrm{loc}}$. Now standard compactness for pseudoholomorphic maps implies that after passing to a subsequence, (u^ν_{01}) converges $\mathcal{C}^\infty_{\mathrm{loc}}$ on $\mathbb{R}\times[0,1/2)$, and as discussed above this implies $\mathcal{C}^\infty_{\mathrm{loc}}$-convergence for the corresponding subsequence of w^ν_0,w^ν_1 near the L_{01}-seam $\mathbb{R}\times\{-1/2\}$.

An analogous argument shows that w^ν_1,w^ν_2 converge $\mathcal{C}^\infty_{\mathrm{loc}}$ near $\mathbb{R}\times\{1/2\}$, so we have now shown that w^ν_0,w^ν_1,w^ν_2 converge $\mathcal{C}^\infty_{\mathrm{loc}}$ everywhere to a $(J^\infty_0(s^\infty,0),J^\infty_1(s^\infty,0),J^\infty_2(s^\infty,0))$-holomorphic figure eight bubble $\underline{w}^\infty$ between L_{01} and L_{12}. The lower gradient bound in (3.27) implies that $\underline{w}^\infty$ is nonconstant and hence has nonzero energy, hence by Lemma 3.2.8 has energy at least $\hbar_8>0$. Finally, rescaling invariance and $\mathcal{C}^\infty_{\mathrm{loc}}$-convergence imply energy concentration of at least $\hbar_8$ at $(s_{N+1},0)$:

$$\liminf_{\nu\to\infty}\int_{B_{\epsilon^\nu}(s^\nu,t^\nu)}\tfrac{1}{2}\bigl|\mathrm{d}\underline{v}^\nu\bigr|^2\geq\liminf_{\nu\to\infty}\sum_{\ell\in\{0,1,2\}}\int_{U^\nu_\ell}w^{\nu*}_\ell\omega_\ell\geq\int_{\mathbb{R}^2}\bigl|\mathrm{d}\underline{w}^\nu\bigr|^2\geq\hbar_8>0.$$

(D02): We will obtain a **squashed eight bubble in $\mathbf{M_{02}}$ with boundary on $\mathbf{L_{01}\circ L_{12}}$** by rescaling

$$w^\nu_\ell(s,t):=v^\nu_\ell\Bigl(s^\nu+\tfrac{s}{R^\nu},\tfrac{\widetilde{f}^\nu(s)}{\widetilde{f}^\nu(0)}\tfrac{t}{R^\nu}\Bigr),$$

where we have set $\widetilde{f}^\nu(s):=R^\nu f^\nu(s^\nu+s/R^\nu)$, to obtain maps

$$\begin{aligned}
w^\nu_0\colon U^\nu_0 &:= \bigl\{(s,t)\mid -R^\nu(\rho+s^\nu)\leq s\leq R^\nu(\rho-s^\nu),\ -\tfrac{\rho R^\nu\widetilde{f}^\nu(0)}{\widetilde{f}^\nu(s)}\leq t\leq-\widetilde{f}^\nu(0)\bigr\}\to M_0,\\
w^\nu_1\colon U^\nu_1 &:= \bigl\{(s,t)\mid -R^\nu(\rho+s^\nu)\leq s\leq R^\nu(\rho-s^\nu),\ -\widetilde{f}^\nu(0)\leq t\leq\widetilde{f}^\nu(0)\bigr\}\to M_1,\\
w^\nu_2\colon U^\nu_2 &:= \bigl\{(s,t)\mid -R^\nu(\rho+s^\nu)\leq s\leq R^\nu(\rho-s^\nu),\ \widetilde{f}^\nu(0)\leq t\leq\tfrac{\rho R^\nu\widetilde{f}^\nu(0)}{\widetilde{f}^\nu(s)}\bigr\}\to M_2.
\end{aligned}$$

Each w^ν_ℓ is pseudoholomorphic with respect to $(\widetilde{J}^\nu_\ell(s,t),j^\nu)$, where the almost complex structure $\widetilde{J}^\nu_\ell(s,t):=J^\nu_\ell(s^\nu+s/R^\nu,\widetilde{f}^\nu(s)t/\widetilde{f}^\nu(0)R^\nu)$ converges $\mathcal{C}^\infty_{\mathrm{loc}}$ to $\widetilde{J}^\infty_\ell:=J^\infty_\ell(s_\infty,0)$ and the symmetric complex structure j^ν on $U^\nu_0\cup U^\nu_1\cup U^\nu_2\subset\mathbb{R}^2$

$$j^\nu(s,t):=\begin{pmatrix}-\frac{t(\widetilde{f}^\nu)'(s)}{\widetilde{f}^\nu(0)} & -\frac{\widetilde{f}^\nu(s)}{\widetilde{f}^\nu(0)}\\ \frac{t^2(\widetilde{f}^\nu)'(s)^2+\widetilde{f}^\nu(0)^2}{\widetilde{f}^\nu(0)\widetilde{f}^\nu(s)} & \frac{t(\widetilde{f}^\nu)'(s)}{\widetilde{f}^\nu(0)}\end{pmatrix}$$

converges $\mathcal{C}^\infty_{\mathrm{loc}}$ to the standard complex structure by Lemma 3.3.3 with $\alpha^\nu:=R^\nu$. The maps w_ℓ^ν also satisfy the Lagrangian seam conditions

$$(w_0^\nu(s,-\widetilde{f}^\nu(0)),w_1^\nu(s,-\widetilde{f}^\nu(0)))\in L_{01},\qquad (w_1^\nu(s,\widetilde{f}^\nu(0)),w_2^\nu(s,\widetilde{f}^\nu(0)))\in L_{12}$$

for all s in $(-R^\nu(\rho+s^\nu),R^\nu(\rho-s^\nu))$. By the $\mathcal{C}^\infty_{\rm loc}$-convergence $\widetilde{f}^\nu(s)/\widetilde{f}^\nu(0)\to 1$ proven in Lemma 3.3.3, we may choose a subsequence and $r^\nu\to\infty$ with $r^\nu\le R^\nu\epsilon^\nu/4$ such that we have

$$\|\widetilde{f}^\nu(s)/\widetilde{f}^\nu(0)\|_{\mathcal{C}^1((-r^\nu,r^\nu))}\le 2 \tag{3.34}$$

for ν sufficiently large. This allows us to translate (3.17) into upper and lower bounds on the energy density $|{\rm d}\underline{w}^\nu|$ for sufficiently large ν,

$$\begin{aligned} |{\rm d}\underline{w}^\nu(0,R^\nu t^\nu)| &\ge \tfrac{1}{2R^\nu}|{\rm d}\underline{v}^\nu(s^\nu,t^\nu)|\ge\tfrac12,\\ \sup_{(s,t)\in B_{r^\nu}(0)}|{\rm d}\underline{w}^\nu(s,t)| &\le \sup_{(s,t)\in B_{\epsilon^\nu}(s^\nu,t^\nu)}\tfrac{4}{R^\nu}|{\rm d}\underline{v}^\nu(s,t)|\le 8.\end{aligned}$$

Here we estimated $\frac{1}{R^\nu}|\partial_s v_\ell^\nu|-\frac{|(\widetilde{f}^\nu)'|}{R^\nu\widetilde{f}^\nu(0)}|\partial_t v_\ell^\nu|\le\left|\partial_s w_\ell^\nu\right|\le\left(\frac{1}{R^\nu}+\frac{|(\widetilde{f}^\nu)'|}{R^\nu\widetilde{f}^\nu(0)}\right)|{\rm d}v_\ell^\nu|$, used the identity $\left|\partial_t w_\ell^\nu\right|=\frac{\widetilde{f}^\nu}{R^\nu\widetilde{f}^\nu(0)}|\partial_t v_\ell^\nu|$, and need to check that $(s,t)\in B_{r^\nu}(0)$ implies $\left(s^\nu+\frac{s}{R^\nu},\frac{\widetilde{f}^\nu(s)}{\widetilde{f}^\nu(0)}\frac{t}{R^\nu}\right)\in B_{\epsilon^\nu}(s^\nu,t^\nu)$ for sufficiently large ν. Indeed, (3.34) yields:

$$\left|\left(\tfrac{s}{R^\nu},\tfrac{\widetilde{f}^\nu(s)}{\widetilde{f}^\nu(0)}\tfrac{t}{R^\nu}-t^\nu\right)\right|/\epsilon^\nu\;\le\;\tfrac{2r^\nu}{R^\nu\epsilon^\nu}+\tfrac{|t^\nu|}{\epsilon^\nu}\;\le\;\tfrac12+\tfrac{\tau_+^\nu+\tau_-^\nu}{2R^\nu\epsilon^\nu}, \tag{3.35}$$

where $R^\nu\epsilon^\nu\to\infty$ and $\tau_\pm^\nu\to\tau_\pm^\infty\in\mathbb{R}$. Next we consider the limiting behaviour of the domain $U_0^\nu\cup U_1^\nu\cup U_2^\nu$. Its straight boundaries diverge $-R^\nu(\rho+s^\nu)\to-\infty$ resp. $R^\nu(\rho-s^\nu)\to\infty$ since $s^\nu\to s^\infty\in(-\rho,\rho)$. The functions $\pm\rho R^\nu\widetilde{f}^\nu(0)/\widetilde{f}^\nu(s)$ of the upper/lower boundary converge $\mathcal{C}^\infty_{\rm loc}$ to ∞ resp. $-\infty$ by Lemma 3.3.3 with $\alpha^\nu:=R^\nu$ and $R^\nu\to\infty$. Finally, the straight seams $\{t=\pm\widetilde{f}^\nu(0)\}$ shrink to a single seam $\{t=0\}$ since we have $\widetilde{f}^\nu(0)=\frac12(\tau_+^\nu-\tau_-^\nu)\to 0$.

Now we may apply Theorem 3.2.9 to this strip shrinking situation to deduce that after passing to a subsequence, $(w_0^\nu(s,t-\widetilde{f}^\nu(0))$ resp. $(w_2^\nu(s,t+\widetilde{f}^\nu(0))$ converge in $\mathcal{C}^\infty_{\rm loc}(-\mathbb{H})$ resp. $\mathcal{C}^\infty_{\rm loc}(\mathbb{H})$ to $\widetilde{J}_0^\infty$- resp. $\widetilde{J}_2^\infty$-holomorphic maps w_0^∞ resp. w_2^∞, and that $(w_1^\nu|_{t=0})$ converges in $\mathcal{C}^\infty_{\rm loc}(\mathbb{R})$ to a smooth map w_1^∞. Furthermore, at least one of w_0^∞,w_2^∞ is nonconstant, and the generalized seam condition $(w_0^\infty(s,0),w_1^\infty(s),w_1^\infty(s),w_2^\infty(s,0))\in L_{01}\times_{M_1}L_{12}$ is satisfied for $s\in\mathbb{R}$, so that (w_0^∞,w_2^∞) is a nonconstant squashed eight bubble with boundary on $L_{01}\circ L_{12}$, with energy bounded below by $\hbar_{L_{01}\circ L_{12}}$. Finally, rescaling invariance, $\mathcal{C}^\infty_{\rm loc}$-convergence, and the containment proven in (3.35) imply energy concentration:

$$\begin{aligned}\liminf_{\nu\to\infty}\int_{B_{\epsilon^\nu}(s^\nu,t^\nu)}\tfrac12|{\rm d}\underline{v}^\nu|^2 &\ge \liminf_{\nu\to\infty}\sum_{\ell\in\{0,2\}}\int_{B_{\epsilon^\nu}(s^\nu,t^\nu)}{v_\ell^\nu}^*\omega_\ell\ge\liminf_{\nu\to\infty}\sum_{\ell\in\{0,2\}}\int_{B_{r^\nu}(0)}{w_\ell^\nu}^*\omega_\ell\\ &\ge\sum_{\ell\in\{0,2\}}\int{w_\ell^\infty}^*\omega_\ell\ge\hbar_{L_{01}\circ L_{12}}.\end{aligned}$$

This ends the construction of a nontrivial bubble in the this last case, (D02), and thus finishes the iterative construction of a subsequence and blow-up points so that (0), (2), and (3) hold. To establish the $\mathcal{C}^\infty_{\rm loc}$-convergence on the complement of the blow-up points claimed in (1) we will apply Theorem 3.2.9 to quilted domains that make up rectangles in $(-\rho,\rho)^2\setminus\{z_1,\ldots,z_N\}$.

Standard elliptic regularity implies that $v_0^\nu(s,t-f^\nu(s))$ resp. $v_2^\nu(s,t+f^\nu(s))$ converge $\mathcal{C}^\infty_{\rm loc}$ on the interior of their domains $(-\rho,\rho)\times(-\rho,0]\setminus Z$ resp. $(-\rho,\rho)\times[0,\rho)\setminus Z$. To extend this convergence to the boundary and to establish convergence of $v_1^\nu(s,0)$, fix a point $(\sigma,0)$ in $(-\rho,\rho)\times\{0\}\setminus Z$, and define three maps by rescaling v_ℓ^ν for $\ell=0,1,2$ and straightening the seams:

$$w_\ell^\nu(s,t) := v_\ell^\nu\big(\sigma+s, \tfrac{f^\nu(s+\sigma)}{f^\nu(\sigma)}t\big).$$

For $r>0$ sufficiently small, these maps form a squiggly strip quilt of size $(f^\nu(\sigma),r)$, which is $(\widetilde{J}_0^\nu,\widetilde{J}_1^\nu,\widetilde{J}_2^\nu,j^\nu)$-holomorphic for $\widetilde{J}_\ell^\nu$ and j^ν the pulled-back almost complex and complex structures

$$\widetilde{J}_\ell^\nu(s,t) := J_\ell^\nu(\sigma+s,\tfrac{f^\nu(s+\sigma)}{f^\nu(\sigma)}t), \qquad j^\nu(s,t) := \begin{pmatrix} -\frac{(f^\nu)'(s+\sigma)}{f^\nu(\sigma)}t & -\frac{f^\nu(s+\sigma)}{f^\nu(\sigma)} \\ \frac{(f^\nu)'(s+\sigma)^2t^2+f^\nu(\sigma)^2}{f^\nu(\sigma)f^\nu(s+\sigma)} & \frac{(f^\nu)'(s+\sigma)}{f^\nu(\sigma)}t \end{pmatrix}.$$

The obedient shrinking $f^\nu \Rightarrow 0$ and the Arzèla–Ascoli theorem guarantee that after passing to a subsequence[12], $f^\nu(s+\sigma)/f^\nu(\sigma)$ converges in $\mathcal{C}^\infty_{\rm loc}$; therefore $\widetilde{J}_\ell^\nu$ and j^ν converge in $\mathcal{C}^\infty_{\rm loc}$ to almost complex and complex structures $\widetilde{J}_\ell^\infty$ and j^∞. As long as r was chosen to be small enough, the bound $\|j^\infty - i\|_{\mathcal{C}^0}\le\epsilon$ holds (where ϵ is the constant appearing in Thm 3.2.9), so Theorem 3.2.9 implies that $w_0^\nu(s,t-f^\nu(\sigma))$, $w_1^\nu(s,0)$, $w_2^\nu(s,t+f^\nu(\sigma))$ converge $\mathcal{C}^\infty_{\rm loc}$ to smooth maps $w_0^\infty, w_1^\infty, w_2^\infty$ that satisfy a generalized seam condition in $L_{01}\times_{M_1}L_{12}$. Since $f^\nu(s+\sigma)/f^\nu(\sigma)$ converges $\mathcal{C}^\infty_{\rm loc}$, we may conclude that $v_0^\nu(s,t-f^\nu(\sigma))$, $v_1^\nu(s,0)$, $v_2^\nu(s,t+f^\nu(\sigma))$ converge $\mathcal{C}^\infty_{\rm loc}$ on a neighborhood of $(\sigma,0)$, and the limit maps satisfy a generalized seam condition in $L_{01}\times_{M_1}L_{12}$. We established convergence away from $(-\rho,\rho)\times\{0\}$ earlier, so we have now proven (1). This finishes the proof of Theorem 3.3.1.

Remark 3.3.4. The purpose of this remark is to discuss the minimal assumptions which allow one to apply Theorem 3.3.1 to symplectic manifolds M_0, M_1, M_2 that are not compact. If the Lagrangian correspondences are compact, then — unlike the "bounded geometry" assumptions in [WeWo1] — we do not explicitly require uniform bounds on metrics and almost complex structures (which were used in [WeWo1] to show energy concentration in a sequence of pseudoholomorphic maps with unbounded gradient). Instead we need to ensure convergence of maps which result from rescaling near a blow-up point of the gradients of a sequence of pseudoholomorphic maps. If the rescaled domains contain boundary or seam conditions, then compactness of the Lagrangians implies $\mathcal{C}^0$-bounds so that the rest of our arguments applies in a precompact neighborhood of the Lagrangians. If the rescaled domains do not contain boundary or seam conditions, or if the Lagrangians in the boundary or seam conditions are noncompact, then $\mathcal{C}^0$-bounds must be obtained a priori from some special properties of the symplectic manifold or Lagrangians.

Note that the $\mathcal{C}^0$-bounds are not merely technical complications — in general, nontrivial parts of sequences of pseudoholomorphic curves can and will escape to infinity, at best yielding punctures and SFT-type buildings in the limit. One way to achieve $\mathcal{C}^0$-bounds would be to work with completed Liouville domains and Lagrangians which are cylindrical

[12]For those choices of (f^ν) that arise from natural geometric situations — e.g. from the figure eight bubble — we expect convergence directly, i.e. without passing to a subsequence.

at infinity, as in Abouzaid–Seidel's definition of the wrapped Fukaya category in [AboSe].

Footnotes 6 and 11 point out the main instances where the specific geometry would have to be considered when dealing with noncompact manifolds. When working with noncompact Lagrangians, one would have to make additional assumptions — such as "bounded geometry" for symplectic manifolds and Lagrangians — to guarantee uniformity of the elliptic estimates in [Bo1].

3.4 Boundary strata and algebraic consequences of strip-shrinking moduli spaces

The purpose of this section is to analyze the expected boundary stratification of strip-shrinking moduli spaces and from this predict the algebraic consequences of figure eight bubbling. While we make an effort to provide convincing arguments for the more surprising features, this part of our exposition will be rather cavalier — aiming only to explain the rough form of what we expect to be able to make rigorous. In particular, all Floer cohomology groups will be considered as ungraded and with coefficients in the Novikov field defined over $\mathbb{Z}_2$, which should be valid as long as sphere bubbling can be avoided. We ultimately expect a fully-fledged graded theory with Novikov coefficients defined over $\mathbb{Q}$ (resp. over $\mathbb{Z}$ in the absence of sphere bubbling).

3.4.1 Boundary stratifications and their algebraic consequences

One of the intuitions in the treatment of pseudoholomorphic curve moduli spaces is that sphere bubbling is "codimension 2" and disk bubbling is "codimension 1". We give a more rigorous statement of this intuition in the polyfold framework and explain its algebraic consequences in Remark 3.4.1 below, and will argue that, in a similarly imprecise sense, figure eight bubbling is "codimension 0" within the "zero-width boundary components" of quilt moduli spaces involving a strip or annulus of varying width.

Remark 3.4.1 (**Codimension and algebraic contribution of sphere and disk bubbles**). In the polyfold setup for pseudoholomorphic curve moduli spaces (whose blueprint is given in [HoWyZe1] at the example of Gromov–Witten moduli spaces), the compactified moduli space is cut out of the ambient polyfold by a (polyfold notion of) Fredholm section, which arises from the Cauchy–Riemann operator. Transversality while preserving compactness can then be achieved by adding a small, compact (possibly multivalued) perturbation, which is supported near the unperturbed moduli space. This equips the perturbed moduli space with the structure of a compact (possibly weighted branched) manifold. For expositions of this theory see e.g. [Ho, HoWyZe2, FaFiGoWe].)

An important feature of the ambient space is that there is a sensible notion of "corner index" — a nonnegative integer associated to each point in the polyfold, so that the points of corner index 0 resp. resp. ≥ 2 should be thought of as the interior resp. smooth part of boundary resp. corner stratification. The transverse perturbation can be chosen compatibly with corner index, so that a Fredholm index 0 section gives rise to a perturbed moduli space lying in the interior of the polyfold, and a Fredholm index 1 section gives rise to a perturbed moduli space whose boundary is given by the intersection of the zero set with the smooth (corner index 1) part of the polyfold boundary. The index 0 components of the perturbed moduli space are then typically used to define an algebraic structure, whose algebraic relations arise from the fact that the Fredholm index 1 component has nullhomologous boundary corresponding to algebraic compositions of Fredholm index 0 contributions. More precisely, the sum over the algebraic contributions of each boundary point is zero, and since these boundary points are given by the zero set of the section restricted to the smooth part of the boundary of the polyfold, the algebraic relations are given by a sum over these boundary strata.

It turns out that interior nodes do not contribute to the corner index, and in particular that curves with sphere bubbles and no other nodes are smooth interior points of the polyfold. This can be understood from the gluing parameters $(R_0, \infty) \times S^1$ used to describe a neighborhood of the node. The corresponding pre-gluing construction provides a local chart for the polyfold, in which the gluing parameters get completed by $\{\infty\}$ to an open disk, which contributes no boundary. On the other hand, gluing at a boundary node or a breaking is described by a parameter in (R_0, ∞), which gets completed by $\{\infty\}$ to a half-open interval, so that pre-gluing in these cases provides local charts in which parameter ∞ indicates a contribution of $+1$ to the corner index. Hence each boundary node (e.g. from disk bubbling), each trajectory breaking (as in Floer theory), and each extra level of buildings (in SFT) contribute 1 to the corner index. This explains why sphere bubbling does not contribute to algebraic relations of the type discussed here, and instead it is the curves with exactly one boundary node (e.g. one disk bubble) or one breaking which contribute to the algebraic relations.

Following the above remark, we need to analyze the boundary stratification of the polyfolds from which the strip-shrinking moduli spaces are cut out in order to predict the algebraic consequences of figure eight bubbling. For that purpose we describe in the following the **pre-gluing constructions that provide the local charts near figure eight and squashed eight bubbles**:

- Gluing a **figure eight** into a pseudoholomorphic quilt has to go along with introducing an extra strip of width $\delta > 0$. Since figure eights do not have an S^1 symmetry, it then remains to fix the length of neck between the bubble and the quilted Floer trajectory. However, this gluing parameter in (R_0, ∞) is in fact fixed by the choice of width $\delta > 0$, as illustrated in Figure 3-1. Hence, while figure eight bubbles can only appear on the $\delta = 0$ boundary, they do not contribute to the corner index. This means that a $\delta = 0$ quilted Floer trajectory with any number of figure eight bubbles will still just have corner index 1. Indeed, the figure eights can only be pre-glued simultaneously since their neck-lengths must all be given by the same strip width $\delta > 0$.

- The configuration of a **squashed eight bubble** with seam in $L_{01} \times_{M_1} L_{12}$ attached to a $\delta = 0$ quilted Floer trajectory has corner index 2 since it can be glued with two independent parameters. Indeed, the pre-gluing construction is to first widen the seam in both the base and the bubble to strips of independent widths $\delta \in [0, 1)$ and $\epsilon \in [0, 1)$ (turning the squashed eight into a figure eight in case $\epsilon > 0$), and to then pre-glue the resulting bubble into the quilt with a gluing parameter $R \in (R_0, \infty]$. Here the strip width δ is determined by (R, ϵ) as follows: Pre-gluing with $R = \infty, \epsilon > 0$ yields a (only approximately holomorphic) figure eight attached to a middle strip of width $\delta(\infty, \epsilon) = 0$ whereas $R < \infty, \epsilon = 0$ produces a (approximately holomorphic) quilted Floer trajectory with middle strip width $\delta(R, 0) = 0$. Positive strip width $\delta(R, \epsilon) > 0$ is achieved only with $R < \infty, \epsilon > 0$, providing the interior of the chart.

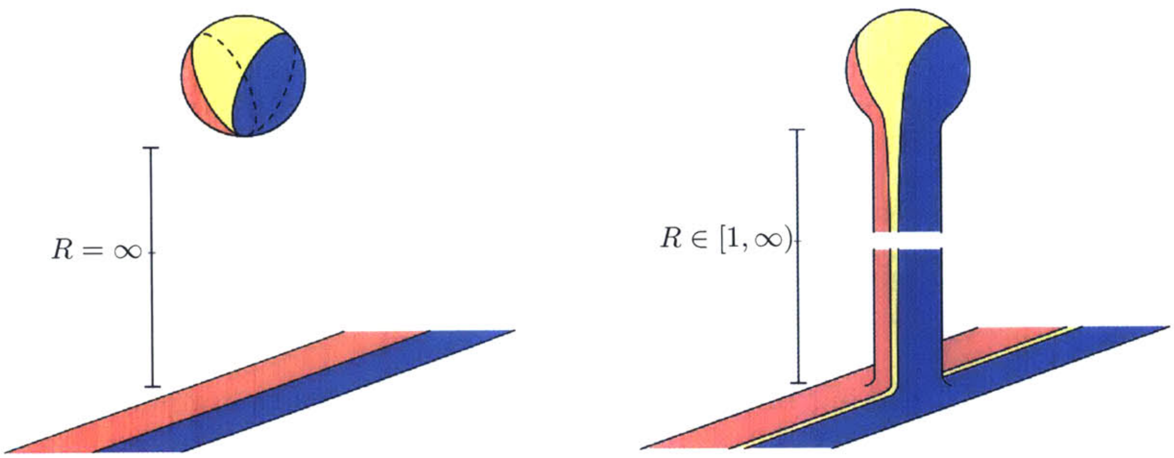


Figure 3-1: A figure eight bubble can be glued to a marked point on a double strip indicated in the left figure. This is done by first fattening the seam in the double strip to a new middle strip of width δ and centered puncture at the old marked point, then overlaying a neighborhood of this puncture with a neighborhood of the figure eight singularity in cylindrical coordinates, and finally interpolating between the maps on the new domain. In this construction the neck-length parameter R is determined by the strip width δ, as illustrated in the right figure: Since the seams are not straight in cylindrical coordinates, the relative shift between bubble and triple strip is determined by the positions of the seams having to match. Note that these figures illustrate the domains of the respective maps, not their images.

- Similarly, a **disk bubble** with boundary in L_{01} or L_{12} attached to a $\delta = 0$ quilted Floer trajectory has corner index 2 since the length of the gluing neck is independent of the width $\delta \geq 0$. In fact, in our tree setup there would be a constant figure eight between the disk and Floer trajectory, so that the gluing parameter is used to pre-glue the disk into the figure eight, and the width parameter pre-glues the resulting figure eight into the Floer trajectory.

Remark 3.4.2 (**Boundary stratification of strip-shrinking moduli spaces**). The gluing construction for squashed eights above indicates that the closures of the two top boundary strata given by $\delta = 0$ quilts with one figure eight bubble (i.e. $R = \infty, \epsilon > 0$) and by $\delta = 0$ quilts with no bubbles (i.e. $R < \infty, \epsilon = 0$) intersect in a corner index 2 stratum consisting of $\delta = 0$ quilts with one squashed eight bubble (i.e. $R = \infty, \epsilon = 0$). To see how a sequence of $\delta = 0$ quilts with one figure eight bubble can converge to a $\delta = 0$ quilt with one squashed eight bubble, note that the moduli space of figure eight bubbles has a boundary stratum in which all energy concentrates at the singularity, so that rescaling yields a squashed eight bubble attached to a constant figure eight.

Note here that different choices of rescaling yield constant figure eights with different values — namely any seam value of the squashed eight except for its value at infinity. Thus the resulting figure eight here is a true **"ghost eight"** in the sense that its value would be determined by the choice of a marked point on the squashed eight; however we do not make a specific choice. In fact, a constant figure eight with no further marked point (except at its singularity, where the squashed eight is attached) would not be stable. We will however include these ghost eights as stable figure eight vertices when describing the bubble trees as colored ribbon trees.

Let us compare this to the fact that a disk bubble with boundary on L_{01} or L_{12} attached (via a constant figure eight) to a $\delta = 0$ quilted Floer trajectory lies in the intersection of a stratum of $\delta > 0$ trajectories with disk bubble and a stratum of $\delta = 0$ trajectories with figure eight bubble. In the first stratum the width goes to zero at the corner, whereas in the second stratum the figure eight converges to a constant figure eight with disk bubble. In this case, however, the constant value of the figure eight is uniquely determined by its attaching point on the quilted Floer trajectory, and the figure eight is stable due to the marked point at which the disk is attached.

3.4.2 Strip shrinking in quilted Floer theory for cleanly-immersed geometric composition

With this framework in place, we now analyze the boundary stratification of a specific strip-shrinking moduli space, from which we will obtain specific algebraic predictions in Section 3.4.4.

The isomorphism between quilted Floer homologies (2.1) under monotone, embedded composition is proven by applying the cobordism argument in Remark 3.4.1 to a moduli space of quilted Floer trajectories with varying width $\delta \in [0, 1]$ of the strip mapping to M_1. Here the boundary arises from the strip widths $\delta = 0$ and $\delta = 1$, since other bubbling or breaking is excluded by the monotonicity assumption. Recall however that this bubble exclusion fails even in monotone cases as soon as the geometric composition is a multiple cover of a smooth Lagrangian (as in many examples of interest, e.g. [We2]). In order to obtain a result that allows for general symplectic manifolds and Lagrangians and cleanly-immersed composition $L_{01} \circ L_{12}$, we need to study the boundary strata of the polyfold which provides an ambient space for a general compactified moduli space of quilted Floer trajectories with varying width. In addition to breaking and bubbling (of disks, squashed eights, and figure eights), the ends of the interval $[0, 1]$ contribute to its corner index. Based on the previous analysis of gluing parameters, we predict that the **top boundary strata of the Gromov-compactified strip-shrinking moduli space** (the strata with corner index 1) are the strata of the following types:

(B1) quilted Floer trajectories for $\delta = 1$;

(B2) once-broken quilted Floer trajectories for $\delta \in (0, 1)$;

(B3) quilted Floer trajectories with one disk bubble on a seam for $\delta \in (0, 1)$;

(B4) quilted Floer trajectories for $\delta = 0$ with generalized seam condition;[13]

(B5) quilted Floer trajectories for $\delta = 0$ with any number of figure eight bubbles.[14]

Within each such boundary component we may also find curves that include trees of sphere bubbles. Furthermore, contributions from (B2) and (B3) necessarily involve curves that for

[13] Since the Lagrangian $L_{01} \circ L_{12}$ is in general just a clean immersion, we require not only that the corresponding seam gets mapped to the composed Lagrangian, but we require this map to lift continuously to $L_{01} \times_{M_1} L_{12}$, and include the lift as data of the Floer trajectory.

[14] In this case, the generalized seam condition requires a lift to $L_{01} \times_{M_1} L_{12}$ that is continuous (and hence smooth) on the complement of the bubbling points, and at each bubbling point is possibly discontinuous in a way that matches with the limits $\lim_{s \to \pm\infty} w_1(s, \cdot)$ of the figure eight.

fixed δ are not cut out transversely, i.e. these contributions come from a finite set of singular values of $\delta \in (0,1)$.

To argue for our prediction, in particular the necessity of allowing figure eight bubbles in (B5), from a more geometric perspective, let us go through the rather silly example of shrinking the strip in standard Floer theory for a pair of Lagrangians $L_{01} \subset \mathrm{pt} \times M_1$ and $L_{12} \subset M_1^- \times \mathrm{pt}$. In this case, the boundary component (B1) represents the Floer differential. The boundary components (B2) and (B3) will be empty, since the Cauchy–Riemann operator for each $\delta > 0$ is just a rescaling of that for $\delta = 1$, and hence all can be made regular simultaneously. Hence the Floer differential must coincide with the algebraic contributions from (B4–5). Indeed, each Floer trajectory can be viewed as a figure eight bubble by Example B.0.7, and in this case is attached to the constant Floer trajectory in $\mathrm{pt} \times \mathrm{pt}$. Broken Floer trajectories are excluded for index reasons. More evidence for the necessity of (B5) are the Floer homology calculations in [We2] between Clifford tori and $\mathbb{RP}^n \subset \mathbb{CP}^n$ resp. the Chekanov torus in $S^2 \times S^2$ using strip shrinking for multiply covered geometric composition, where bubbling can only be excluded for classes of Floer trajectories whose limits are not self-connecting. Nonzero results for the corresponding entries of the differential from other calculation methods then indirectly show nontrivial figure eight contributions.

At this point, a reader comfortable with evaluation maps into appropriate spaces of chains may skip to the algebraic consequences in Section 3.4.4. However, we will construct these algebraic structures from the following more complicated moduli spaces that will simplify both our analytic and algebraic work, and also serve to further solidify our prediction of boundary stratifications.

3.4.3 Morse bubble trees arising from strip-shrinking moduli spaces

To capture the algebraic effect of figure eight bubbling in terms of Morse chains on $L_{01} \times_{M_1} L_{12}$, we will extend strip-shrinking moduli spaces by allowing Morse flow lines between figure eight, squashed eight, and disk bubbles[15]. This will be achieved by organizing the tree of bubble vertices and Morse edges into a colored metric ribbon tree (as introduced in [MaWo, Def. 7.1]) whose root is the base quilt in which the strip was being shrunk. This approach is analogous to constructing the A_∞-algebra of a single Lagrangian submanifold via trees of disk vertices and Morse edges[16] which we sketch in the following Remark before explaining its generalization to strip-shrinking moduli spaces.

Remark 3.4.3 (**Polyfold setup and boundary stratification for trees of disks with Morse edges**). Consider a single disk bubble attached by a marked point to e.g. a Floer trajectory. If we enlarge the moduli space by Morse flow lines between nodal pairs of boundary marked points, then boundary strata with length 0 flow lines cancel strata with boundary nodes and we obtain a compactified moduli space of metric ribbon trees whose root is the attaching point, vertices are represented by pseudoholomorphic disks (modulo appropriate reparametrizations which we will not discuss), and edges are represented by generalized Morse trajectories (including broken trajectories that compactify the space of finite length Morse trajectories) that are directed toward the root.

[15] Here we identify quilted spheres with two patches in M_k and M_{k+1} with disks in $M_k^- \times M_{k+1}$; see footnote 1.

[16] This has been a partially realized vision in the field for a while. Formally, it follows the A_∞ perturbation lemma [Se, Prop. 1.12] for transferring an A_∞-structure on a space of differential chains to the space of Morse chains. A related moduli space setup was proposed in [CoLa] but with a different algebraic goal.

In [LiWe], assuming the absence of sphere bubbling, this moduli space is described as the zero set of a Fredholm section in an M-polyfold bundle. This section is given by the Cauchy–Riemann operators on each vertex together with the matching conditions for each edge between the endpoints of the Morse trajectory and corresponding marked point evaluations of the disk maps. The ambient M-polyfold is the space of trees in which vertices are represented by (reparametrization equivalence classes of) not necessarily pseudoholomorphic maps and edges are represented by generalized Morse trajectories. Nodal configurations with edge length 0 are interior points of this space by pre-gluing of the nodal disks into a single vertex (this is made rigorous in terms of M-polyfold charts arising from the pre-gluing construction). Hence the boundary stratification of this space is induced by the compactified space of Morse trajectories — which was given a smooth structure in [We4], with corner index equal to the number of critical points at which a trajectory breaks.

Now the arguments of Remark 3.4.1 yield an algebraic structure from counting isolated solutions whose relations are given by summing over the top boundary strata (those with corner index 1). In this case, adding incoming Morse edges from input critical points yields a curved A_∞-algebra because the top boundary strata — configurations with exactly one broken trajectory, i.e. edge of length ∞ — correspond to the top boundary strata of a space of metric ribbon trees. The stable trees in the latter realize Stasheff's associahedra, so that the boundary strata yield the A_∞-relations with the exception of terms involving μ^1 or μ^0. These additional terms arise from breaking into two subtrees of which one is unstable with zero or one incoming Morse edge. Similarly, considering Floer trajectories for pairs of Lagrangians (or quilted Floer trajectories) with several bubble trees (on each boundary component resp. seam) yields the relations for the Floer differential coupled with the A_∞-algebras of the Lagrangians.

Sphere bubbling can be included here by extending the ambient space of disk maps and Morse edges to allow for trees of spheres attached to the maps. This introduces the additional complication of isotropy, turning the ambient space into a polyfold (M-polyfolds are a special case with trivial isotropy) and forces the use of multivalued perturbations, thus yielding rational counts. However, as discussed in Remark 3.4.1, this does not affect the boundary stratification and algebraic consequences.

Ignoring sphere bubbling as above, we introduce Morse flow lines into strip-shrinking moduli spaces in two stages: First, we allow for Morse edges between disk bubbles and the domain in which they occurred. As in Remark 3.4.3 this captures disk bubbling on seams in an algebraic coupling with the curved A_∞-algebras generated by Morse chains on the Lagrangian correspondences that are not involved in the strip shrinking. If such a tree of disks was attached to a seam in the quilt for $\delta > 0$, then in the $\delta = 0$ limit we represent it by a tree attached to the respective seam of a constant figure eight bubble. We then begin the second stage of extending the strip-shrinking moduli space by introducing Morse flow lines on $L_{01} \times_{M_1} L_{12}$ between the quilt and figure eights. Since a zero width strip with any number of figure eight bubbles is a corner index 1 boundary point, we need to extend with a single normal parameter, so take all Morse edges of the same length $-\delta$. This extends the strip width parameter from $\delta \in [0,1]$ to $\delta < 0$, and to compactify the resulting extended moduli space we allow for Morse breaking (simultaneously for each figure eight) as $\delta \to \infty$ but also need to take into account that besides disk bubbling (which is dealt with as before) we may have squashed eight bubbles appearing by energy concentration on the base quilt or at the singularity of a figure eight bubble (as in Remark 3.4.2, with the remaining figure eight being either nonconstant or a ghost eight). We cancel these boundary components again by introducing Morse flow lines on $L_{01} \times_{M_1} L_{12}$, whose length can now vary individually. However, since squashed eights appear between the base quilt and figure eights, the length of these Morse edges has to be accounted for in the condition of figure eights all being at the same "Morse distance" from the base quilt. So the bubble hierarchy indicated in Figure 1-2 yields a construction of the extended moduli space (more precisely its part on which $\delta \in [-\infty, 0]$): It consists of **Morse bubble trees over quilts of strip width zero** that are organized into colored metric ribbon trees (see [MaWo, Def. 7.1]) as follows:

- The root is represented by the base quilt in which the strip has been shrunk to width 0.
- Other vertices are represented by a pseudoholomorphic disk, squashed eight, figure eight, or ghost eight. Figure eights and ghost eights are the colored vertices, of which there is exactly one between each leaf and the root. The squashed eights are exactly the vertices between the root and a colored vertex.
- Each edge is labeled by a "Morse length" in $[0, \infty]$, and the "Morse distance" between each colored vertex and the base quilt (the sum of lengths of edges in between base quilt and figure eight resp. ghost eight) is the same. We denote this "figure eight height" by $-\delta$ for $\delta \in [-\infty, 0]$.
- Each edge attached to a disk vertex is represented by a generalized Morse trajectory on L_{01} resp. L_{12} of the given length. Each edge between figure eights, squashed eights, and the root is represented by a generalized Morse trajectory on $L_{01} \times_{M_1} L_{12}$ of the given length.
- Disks and squashed eights are constant only if the vertex has valence ≥ 3. Figure eights are constant only if the vertex has valence ≥ 2. Ghost eights only appear as colored leaf attached by a ghost edge to a squashed eight vertex of valence 2. The ghost vertex and ghost edge carry no geometric information other than a length of the ghost edge in $[0, \infty]$.

∘ Later on we generalize these moduli spaces further to obtain algebraic coupling with the Morse chain complexes on L_{01} and L_{12}. For that purpose we allow "half-infinite edges" which terminate at a leaf and are represented by generalized Morse trajectories in the compactifications $\overline{\mathcal{M}}(x_-, L_{ij})$ of unstable manifolds of critical points x_-.

Note that constant figure eights are stable from an isotropy point of view when their vertex has valence ≥ 2. The ghost eight leaves are forced by the fact from Remark 3.4.2 that the moduli space (resp. ambient polyfold) of figure eight quilts has a boundary stratum on which all energy concentrates at the singularity, so that rescaling yields a squashed eight bubble that is attached to a constant figure eight quilt, whose value can be varied by changing the rescaling (that's what we denote by ghost eight). If the figure eight was nonconstant, then the resulting boundary stratum of the moduli space of fixed tree type is cancelled by the boundary stratum in which the Morse edge between a squashed eight and a figure eight vertex is of length 0. Our setup including ghost eight leaves is equivalent to requiring squashed eight vertices to have "Morse distance" from the base quilt less or equal to the "figure eight height", and using the boundary strata resulting from equality to cancel the above mentioned strata in which a figure eight vertex degenerates into a squashed eight. From here we can note that any breaking of trajectories between squashed eights or figure eights and the main component corresponds to Morse height $-\delta \to \infty$ and thus forces at least one breaking between each figure eight vertex and the root. (Between ghost eights and

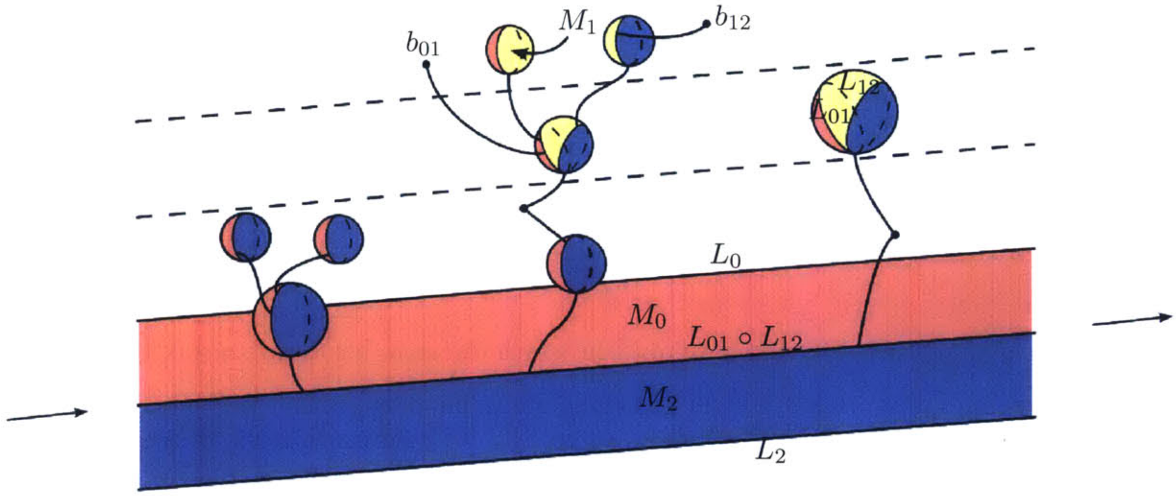


Figure 3-2: A contribution to the differential on $CF(L_0, (L_{01} \circ L_{12}, \sum_{k,l \geq 0} b_{02}^{k|l}), L_2)$. The two subtrees above the Morse critical points contribute to $b_{02}^{1|1}$ and $b_{02}^{0|0}$, respectively. The dashed lines indicate the level structure of the colored metric ribbon tree. Additional half-infinite edges are labeled with the Morse cochains b_{01} or b_{12}, indicating a formal sum of trees whose half-infinite edges are Morse edges starting at the Morse critical points that represent the cochains.

the root, this can be achieved by a broken Morse trajectory between a squashed eight and the root, or by a ghost edge of length ∞.) If there is just one breaking for each figure eight (and for each ghost eight either the ghost edge is infinite or there is one breaking between the squashed eight and quilt), then the result lies in the top boundary stratum, but any additional breaking of an edge adds to the corner index individually.[17]

As in Remark 3.4.3, we expect to obtain an ambient polyfold (or M-polyfold if sphere bubbling can be a priori excluded) by replacing the pseudoholomorphic curves and quilts with appropriate spaces of not necessarily pseudoholomorphic maps modulo reparametrization. Making this rigorous will require a precise setup of pre-gluing constructions for squashed eights and figure eights as M-polyfold charts from [BoWe1], which will also make the predicted boundary stratification rigorous. Further steps in the program are the Fredholm property of the Cauchy–Riemann operator, and formal setups for the construction of gluing-coherent perturbations and orientations. However, we can already see that the boundary stratification of this polyfold is — apart from strata of types (B1–4) on page 90 — induced by the boundary structure of the compactified Morse trajectory spaces from [We4]. These $\delta = -\infty$ boundary components correspond to the boundary strata of the compactified space of colored metric ribbon trees (given by allowing infinite edge length). Since stable trees of this sort realize Stasheff's multiplihedra [MaWo, Thms. 1.1, 7.6], we expect to obtain A_∞-functor relations from this boundary component. More precisely, we expect to obtain algebraic relations from the **top boundary strata of the compactified strip-shrinking moduli space with Morse bubble trees**, which replace our previous list (B1–5) as follows:

[17]If there are N figure eight vertices, then this effect is analogous to the breaking of finite length Morse trajectories in the N-fold Cartesian product of $L_{01} \times_{M_1} L_{12}$: The first breaking has to happen simultaneously in all components since their lengths are coupled; further breakings are independent since trajectories can be constant in various components.

(B1') quilted Floer trajectories for $\delta = 1$ which may include trees of disk bubbles with finite length Morse edges;

(B2') once-broken quilted Floer trajectories for $\delta \in (0,1)$ which may include trees of disk bubbles with finite-length Morse edges;

(B2") once-broken quilted Floer trajectories can also appear for $\delta \in (-\infty, 0)$, where they consist of Floer trajectories of width 0 with a colored tree of figure eight height $-\delta$, and may include further trees of disk bubbles with finite-length Morse edges;

(B3') quilted Floer trajectories with one disk bubble on a seam for $\delta \in (0,1)$ are canceled as boundary components, but new boundary components contain quilted Floer trajectories for $\delta \in (0,1)$ with trees of disk bubbles, in which one Morse edge is broken once;

(B3") quilted Floer trajectories of width 0 with a single broken Morse edge can also appear for $\delta \in (-\infty, 0)$, i.e. in a colored tree of figure eight height $-\delta$ with the broken edge occurring either above the figure eight height or in a tree of disk bubbles on another seam;

(B4') quilted Floer trajectories for $\delta = 0$ may include trees of disk bubbles with finite length Morse edges attached to seams that are not involved in the shrinking; but when the width goes to zero in the presence of a tree of disk bubbles on L_{01} or L_{12}, this is viewed as constant figure eight to which this tree is attached, and is canceled like other strata of type (B5);

(B4") quilted Floer trajectories for $\delta = 0$ with squashed eight bubbles are canceled as boundary components, but new boundary components contain quilted Floer trajectories of width 0 with a colored tree of figure eight height $-\delta = \infty$, all of whose colored vertices are ghost eights with infinite ghost edges; equivalently, these are quilted Floer trajectories for $\delta = 0$ with trees of squashed eights and finite Morse edges attached to the shrunk seam;

(B5') quilted Floer trajectories for $\delta = 0$ with figure eight bubbles are canceled as boundary component, but new boundary components contain quilted Floer trajectories of width 0 with a colored tree of figure eight height $-\delta = \infty$, that is between each figure eight or ghost eight and main quilt there is exactly one infinite edge — either an infinite ghost edge or a once-broken Morse trajectory (of which there is at least one, otherwise this component is listed under (B4")) — and all other edges have finite length.

An example of a boundary point of type (B5') is given in Figure 3-2, where the dashed lines indicate the level structure of the colored metric ribbon tree resulting from strip shrinking: In the first level above the root quilt, all vertices are represented by squashed eights, whereas in the third level the vertices are represented by disk bubbles with boundary on L_{01} or L_{12}. The second level provides the division since between each leaf and the root there is exactly one vertex represented by a figure eight — though we did not graphically represent the ghost eight vertices above the two squashed eight leaves on the left subtree. The figure eight height of this tree is ∞, reflected by at least one broken trajectory below each figure eight — in particular, the graphically unrepresented ghost edges between the squashed eight leaves and ghost figure eight vertices have length ∞. The corner index is

1 since there is exactly one breaking for each figure eight (resp. infinite edge for the ghost eights). A similar colored tree of corner index 1 and figure eight height ∞ could be obtained by giving the ghost edges above the squashed eight leaves finite lengths, but replacing either both edges below the leaves with a broken trajectory, or having just one broken trajectory at the edge which attaches both leaves to the root. However, we do not expect algebraic contributions from the latter tree types: Geometrically, this would mean an isolated solution on the boundary of the polyfold containing the moduli space of figure eights with outgoing infinite Morse edge. In terms of our tree setup, such solutions aren't isolated since the length of the finite ghost edge can be varied.

3.4.4 Floer homology isomorphism for general cleanly-immersed geometric composition

To generalize the isomorphism between quilted Floer homologies (2.1) under monotone, embedded composition to general symplectic manifolds and Lagrangians and cleanly-immersed composition $L_{01} \circ L_{12}$, we analyzed in Section 3.4.1 the boundary strata of the polyfold which provides an ambient space for a general compactified moduli space of quilted Floer trajectories with varying width. The cobordism argument outlined in Remark 3.4.1 then predicts an algebraic identity from summing over the boundary strata (B1–5) on page 90 resp. the refined boundary strata on page 94.

We expect the strata of types (B2), (B3) resp. (B2'), (B2"), (B3'), (B3") to appear only at finitely many singular values of strip width $\delta \in (0,1)$ or figure eight height $\delta \in (-\infty, 0)$ and to provide a chain homotopy equivalence between two Floer complexes: The first is in both frameworks defined from the regular strip width $\delta = 1$, with the differential given by counting solutions of type (B1) resp. (B1'). In the Morse framework, the second Floer complex is defined from counting regular solutions of types (B4'), (B4"), and (B5'). In the framework of page 90, the second complex should arise from solutions of types (B4) and (B5) at $\delta = 0$, though it is unclear in what sense the latter might be made regular.

Up to such a chain homotopy equivalence, or assuming there are no singular values in $(-\infty, 1)$, we obtain the following identity relating the Floer differential $\mu^1_{(\delta=1)}$ arising from strip width $\delta = 1$ and the Floer differential $\mu^{1|0}_{(\delta=0)}$ arising from strip width $\delta = 0$ with generalized seam condition in $L_{01} \times_{M_1} L_{12}$:[18]

$$\mu^1_{(\delta=1)}(-) = \sum_{k \geq 0} \mu^{1|k}_{(\delta=0)}(-\mid b_{02}, \ldots, b_{02}). \tag{3.36}$$

In the Morse framework, the moduli spaces defining the differentials $\mu^1_{(\delta=1)}$ and $\mu^{1|0}_{(\delta=0)}$ both allow for trees of disks with finite Morse edges (including trees of squashed eights attached to the seam obtained from strip shrinking). Figure eight bubbling is in both frameworks encoded in the higher operations $\mu^{1|k}_{(\delta=0)}$ for $k \geq 1$. In the framework of (B5), this operation should be defined from quilted Floer trajectories of middle strip width 0 with k incoming marked points on the seam labeled by the immersion $L_{01} \circ L_{12}$, and b_{02} should be a chain

[18] Note that the moduli spaces with generalized seam condition involve a choice of lift of the seam values to $L_{01} \times_{M_1} L_{12}$. So in case $L_{01} \circ L_{12}$ is a smooth Lagrangian correspondence albeit multiply covered by $L_{01} \times_{M_1} L_{12}$, this Floer complex is generated by lifts of intersection points. The differential $\mu^{1|0}_{(\delta=0)}$ only counts Floer trajectories with smooth seam lift, whereas the terms $\mu^{1|k}_{(\delta=0)}$ for $k \geq 1$ will allow for jumps in the seam lift.

obtained from a moduli space of figure eight bubbles by evaluation at the singularity. In the absence of an approach for making the latter rigorous, we will construct the operations[19]

$$\mu^{1|k}_{(\delta=0)} : CF(L_0, L_{01} \circ L_{12}, L_2) \otimes CM(L_{01} \times_{M_1} L_{12})^{\otimes k} \longrightarrow CF(L_0, L_{01} \circ L_{12}, L_2)$$

in the Morse framework from quilted Floer trajectories of strip width 0 with k incoming Morse edges (represented by half-infinite trajectories in $L_{01} \times_{M_1} L_{12}$ starting at a Morse critical point) attached (possibly via trees of squashed eights and finite Morse edges) to the middle seam. Then (B5') indicates that the Morse cochain

$$b_{02} \in CF(L_{01} \circ L_{12}, L_{01} \circ L_{12}) := CM(L_{01} \times_{M_1} L_{12}) \tag{3.37}$$

should be defined by counting regular isolated figure eight bubble trees as follows:

- Exactly one vertex is represented by a figure eight, and this vertex lies between every leaf and the root. All vertices between leaves and the figure eight are represented by pseudo-holomorphic disks, and all vertices between the figure eight and the root are represented by squashed eights.
- Each edge attached to a disk vertex is represented by a finite Morse trajectory on L_{01} resp. L_{12}, and all other edges are represented by a finite Morse trajectory on $L_{01} \times_{M_1} L_{12}$.
- The root vertex is represented by a figure eight or squashed eight with a marked point at the singularity, to which an outgoing Morse edge is attached, i.e. a half-infinite trajectory in $L_{01} \times_{M_1} L_{12}$ ending at a Morse critical point.
- Disks and squashed eights are constant only if the vertex has valence ≥ 3. Figure eights are constant only if the vertex has valence ≥ 2.

Once such operations are defined, (3.36) identifies (up to chain homotopy equivalence) the quilted Floer chain complexes

$$CF\big(L_0, L_{01}, L_{12}, L_2\big) \simeq CF\big(L_0, (L_{01} \circ L_{12}, b_{02}), L_2\big),$$

where the differential on the left hand side is $\mu^1_{(\delta=1)}$ and the differential on the right hand side is the twisted differential $\partial_{b_{02}} := \sum_{k\geq 0} \mu^{1|k}_{(\delta=0)}(-\,|\, b_{02}, \ldots, b_{02})$. Here the right hand side treats $L_{01} \circ L_{12}$ as an immersion. If this is an embedding then the right hand side is the Floer chain complex of the Lagrangian $L_{01} \circ L_{12} \subset M_0^- \times M_2$ twisted by the Morse cochain b_{02}. An example of a contribution to the twisted Floer differential is Figure 3-2 without the middle tree. This result is meaningful if the cyclic Lagrangian correspondence L_0, L_{01}, L_{12}, L_2 is naturally unobstructed in the sense that the differential $\partial = \mu^1_{(\delta=1)}$ satisfies $\partial^2 = 0$. In particular, it asserts that the twisted differential on the right hand side satisfies $\partial^2_{b_{02}} = 0$. To understand more intrinsically why the twisted differential squares to zero, we need to go into the A_∞ algebra.

Remark on $\mathbf{A_\infty}$ terminology: *In the upcoming sections, we will denote by $\big(\mu^d_{01}\big)_{d\geq 0}$, $\big(\mu^d_{12}\big)_{d\geq 0}$, resp. $\big(\mu^d_{02}\big)_{d\geq 0}$ the curved A_∞-algebras associated to L_{01}, L_{12}, resp. $L_{01} \circ L_{12}$, constructed on Morse chain complexes as outlined in Remark 3.4.3. If working with the*

[19] When the composition $L_{01} \circ L_{12}$ is embedded, we expect $\mu^{1|k}_{(\delta=0)}$ to agree with the A_∞-structure map $CF(L_{01} \circ L_{12}, L_0 \times L_2) \otimes CF(L_{01} \circ L_{12}, L_{01} \circ L_{12})^{\otimes k} \to CF(L_{01} \circ L_{12}, L_0 \times L_2)$ on $\mathrm{Fuk}(M_0^- \times M_2)$.

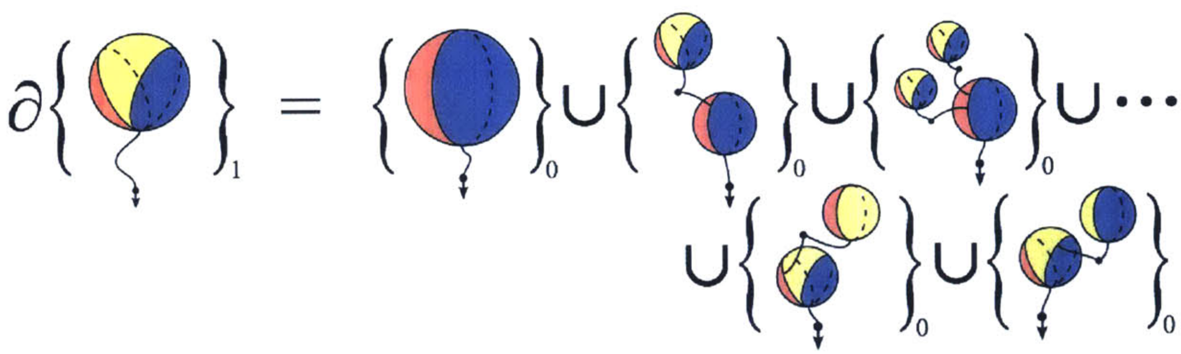


Figure 3-3: The expected boundary strata of a figure eight moduli space explain various algebraic identities in Remark 3.4.4. If contributions from disk bubbles on L_{01}, L_{12} can be excluded (i.e. $\mu_{01}^0 = \mu_{12}^0 = 0$) then the element $b_{02} \in CM(L_{01} \times_{M_1} L_{12})$ should solve the Maurer–Cartan equation for $L_{01} \circ L_{12}$. For monotone, embedded composition, the figure eight bubbles can be excluded to explain the identity $n_{L_{01}} + n_{L_{12}} = n_{L_{01} \circ L_{12}}$ between disk counts.

latter, we will usually assume that $L_{01} \circ L_{12}$ *is embedded, though there are extensions to multiply covered and even cleanly immersed cases, as outlined in Remark 1.2.2.*

Moreover, we will call $b \in CF(L, L)$ *a* bounding cochain for the Lagrangian L *if it satisfies the Maurer–Cartan equation* $\sum_{d \geq 0} \mu^d(b, \dots, b) = 0$. *When a quilted Floer differential is twisted by bounding cochains for each Lagrangian correspondence, it will square to zero. It is however also possible that twisting with more general cochains yields a chain complex.*

Remark 3.4.4. The vanishing $\partial_{b_{02}}^2 = 0$ generally follows from the identification of differentials $\partial = \partial_{b_{02}}$ in (3.36) together with the assumption $\partial^2 = 0$. In more special cases we expect this to be a consequence of $b_{02} \in CF(L_{01} \circ L_{12}, L_{01} \circ L_{12})$ being a bounding cochain, i.e. satisfying the Maurer–Cartan equation $\sum_{d=0}^{\infty} \mu_{02}^d(b_{02}, \dots, b_{02}) = 0$. (Here we assume $L_{01} \circ L_{12}$ to be an embedded composition, though this remark should extend to the cleanly-immersed setting.) This should follow from a cobordism argument illustrated in Figure 3.4.4: Consider the 1-dimensional moduli space of figure eight quilts between L_{01} and L_{12}, with a half-infinite outgoing Morse trajectory on $L_{01} \times_{M_1} L_{12}$ attached to its singularity. Extrapolating from the boundary analysis in §§3.4.2–3.4.3, we expect the 0-dimensional boundary strata to come in two types:

- Some strata are given by squashed eights with seam in $L_{01} \times_{M_1} L_{12}$, with one outgoing half-infinite Morse trajectory on $L_{01} \times_{M_1} L_{12}$ attached to the singularity, and $d \geq 0$ figure eight bubbles attached to the seam via once-broken Morse trajectories on L_{01}. The formal sum over the limiting critical points of the outgoing trajectories of such isolated solutions yields $\mu_{02}^d(b_{02}, \dots, b_{02})$.

- The remaining strata are given by figure eights with one outgoing half-infinite Morse trajectory attached to the singularity, and a disk bubble mapping to $(M_k^- \times M_{k+1}, L_{k(k+1)})$ for either $k = 0$ or $k = 1$ attached to the $L_{k(k+1)}$-seam via a once-broken Morse trajectory on $L_{k(k+1)}$. The formal sum over the limiting critical points of such isolated solutions yields $C^2(|\mu_{01}^0)$ resp. $C^2(\mu_{12}^0|)$ when $k = 0$ resp. $k = 1$, where C^2 is the curved A_∞-bifunctor whose blueprint we sketched in Chapter 1.

As boundaries of a 1-dimensional moduli space, the algebraic contributions of these boundary strata should sum to zero. (In fact, this equation is also a formal consequence of the curved A_∞-bifunctor relations satisfied by C^2.) In the special case $\mu^0_{01} = \mu^0_{12} = 0$ (i.e. when there are no disk bubbles on L_{01} or L_{12}, or their contributions cancel) this yields the expected Maurer–Cartan equation $\sum_{d\geq 0}\mu^d_{02}(b_{02},\ldots,b_{02}) = 0$.

This also illuminates an identity between disk counts noted in [WeWo1, Remark 2.2.3]: Working with monotone orientable Lagrangians and embedded composition, one expects both differentials $\partial_\delta := \mu^1_{(\delta>0)}$ and $\partial_0 := \mu^1_{(\delta=0)}$ to square to multiples of the identity, $\partial_\delta^2 = w_\delta \mathrm{id}$ resp. $\partial_0^2 = w_0\mathrm{id}$, with $w_\delta = n_{L_0} + n_{L_{01}} + n_{L_{12}} + n_{L_2}$ resp. $w_0 = n_{L_0} + n_{L_{01}\circ L_{12}} + n_{L_2}$ given by sums of counts n_L of Maslov index 2 disks through a generic point on the Lagrangian L. Arguing by strip shrinking identifying the differentials, [WeWo1] concluded $w_\delta = w_0$ and hence $n_{L_{01}} + n_{L_{12}} = n_{L_{01}\circ L_{12}}$. This identity can now also be seen directly from the above cobordism argument: Monotonicity excludes nonconstant figure eight bubbles, which reduces the boundary strata on the right hand side of Figure 3.4.4 to the first and last two types, corresponding to $n_{L_{01}\circ L_{12}}$ and $n_{L_{01}}$, $n_{L_{12}}$, respectively.

Next, we relax the unobstructedness to the assumption that the Lagrangians L_0, L_{01}, L_{12}, L_2 are equipped with cochains $\underline{b} = (b_0, b_{01}, b_{12}, b_2)$ so that the twisted differential $\partial_{\underline{b}}$ (which arises from adding marked points labeled with b_0, b_{01}, b_{12}, resp. b_2 to the $\delta = 1$ quilted Floer trajectories) satisfies $\partial_{\underline{b}}^2 = 0$. Then we may add these cochains to the previous strip-shrinking moduli space as incoming Morse edges whose starting points represent b_0, b_{01}, b_{12}, resp. b_2 and whose endpoints correspond to marked points on the seams labeled L_0, L_{01}, L_{12}, resp. L_2 anywhere on the quilted Floer trajectory or the attached bubble trees. Then an analogous cobordism argument yields a chain homotopy equivalence

$$CF\big((L_0,b_0),(L_{01},b_{01}),(L_{12},b_{12}),(L_2,b_2)\big) \;\simeq\; CF\big((L_0,b_0),\big(L_{01}\circ L_{12},\textstyle\sum_{k,\ell\geq 0} b_{02}^{k|\ell}\big),(L_2,b_2)\big),$$

where the Morse cochains $b_{02}^{k|\ell} \in CF(L_{01}\circ L_{12}, L_{01}\circ L_{12})$ are obtained by adding incoming Morse edges to the figure eight bubble trees that define $b_{02} =: b_{02}^{0|0}$. More precisely, we attach k incoming Morse edges representing $b_{12} \in CF(L_{12}, L_{12})$ to the L_{12} seams anywhere on the bubble tree, and we attach ℓ incoming Morse edges representing b_{01} to the L_{01} seams. Figure 3-2 provides an example of figure eight contributions to the twisted Floer differential for the composed Lagrangian correspondences on the right-hand side of the equivalence.

This demonstrates, as advertised in the introduction, that the isomorphism of quilted Floer homologies (2.1) should generalize in a straightforward fashion to the nonmonotone, cleanly-immersed case as isomorphism of quilted Floer homologies with twisted differentials,

$$HF\big(\ldots,(L_{01},b_{01}),(L_{12},b_{12}),\ldots\big) \;\simeq\; HF\big(\ldots,(L_{01}\circ L_{12}, 8(b_{01},b_{12})),\ldots\big),$$

in which the cochain $8(b_{01}, b_{12})$ for the composed Lagrangian is obtained from moduli spaces of figure eight bubble trees with inputs b_{01} and b_{12}. In particular, the cochain $8(0,0) = b_{02}$ in (3.37) for vanishing inputs is a generally nonzero count of figure eight bubbles.

Remark 3.4.5. The cobordism argument in Remark 3.4.4 can be adapated to the situation that L_{01}, L_{12} are equipped with bounding cochains $b_{01} \in CF(L_{01}, L_{01})$, $b_{12} \in CF(L_{12}, L_{12})$: This time, for every $k, \ell \geq 0$ we consider 1-dimensional moduli spaces of figure eight quilts with one outgoing half-infinite Morse trajectory attached to the singularity, and k resp. ℓ incoming half-infinite Morse trajectories attached to the L_{01}-seam resp. the L_{12}-seam. The algebraic contributions of the boundary strata with incoming Morse critical points representing b_{01} and b_{12} should sum to zero. Summing over all $k, \ell \geq 0$, we obtain the expected equation

$$\sum_{k,\ell\geq 0} \sum_{\substack{k_1+\cdots+k_d=k,\\ \ell_1+\cdots+\ell_d=\ell}} \mu_{02}^d(b_{02}^{k_d|\ell_d}, \ldots, b_{02}^{k_1|\ell_1})$$

$$= \sum_{k,\ell\geq 0} \sum_{a+d\leq k} C^2\Big(\underbrace{b_{12}, \ldots, b_{12}}_{\ell} \mid \underbrace{b_{01}, \ldots, b_{01}}_{k-a-d}, \mu_{01}^d(b_{01}, \ldots, b_{01}), \underbrace{b_{01}, \ldots, b_{01}}_{a}\Big)$$

$$+ \sum_{k,\ell\geq 0} \sum_{a+d\leq \ell} C^2\Big(\underbrace{b_{12}, \ldots, b_{12}}_{\ell-a-d}, \mu_{12}^d(b_{12}, \ldots, b_{12}), \underbrace{b_{12}, \ldots, b_{12}}_{a} \mid \underbrace{b_{01}, \ldots, b_{01}}_{k}\Big).$$

The right side vanishes, after a reorganization, by the Maurer–Cartan equations for b_{01} and b_{12}. Then a reorganization of the left side yields the expected Maurer–Cartan equation for $L_{01} \circ L_{12}$,

$$\sum_{d\geq 0} \mu_{02}^d\Big(\textstyle\sum_{k,\ell\geq 0} b_{02}^{k|\ell}, \ldots, \sum_{k,\ell\geq 0} b_{02}^{k|\ell}\Big) = 0.$$

Appendix A

Removal of singularity for cleanly intersecting Lagrangians

In this appendix, we sketch a proof of removal of singularity for a holomorphic curve satisfying a generalized Lagrangian boundary condition in an immersed Lagrangian with locally-clean self-intersection. We emphasize that this is not a new result, see e.g. [Abb, CiEkLa, Fr, IvSh, Sc]. We have included the following proposition in this paper because our methods allow us to give a short proof.

This removal of singularity will be stated for maps u with Lagrangian boundary conditions lifting to paths γ, γ':

$$\begin{gathered} u\colon (B(0,1)\cap \mathbb{H})\setminus\{0\} \to M, \qquad \gamma'\colon (-1,0) \to L', \qquad \gamma\colon (0,1)\to L, \\ \varphi'(\gamma'(s')) = u(s',0), \qquad \varphi(\gamma(s)) = u(s,0) \qquad \forall\, s' \in (-1,0),\ s\in(0,1), \\ \partial_s u + J(s,t,u)\partial_t u = 0, \qquad E(u) := \textstyle\int u^*\omega < \infty, \end{gathered} \tag{A.1}$$

where (M,ω) is a closed symplectic manifold, $\varphi\colon L \to M$ and $\varphi'\colon L' \to M'$ are Lagrangian immersions with L, L' closed, and J is an almost complex structure $J\colon B(0,1)\cap\mathbb{H} \to \mathcal{J}(M,\omega)$. We will assume that $\varphi(L), \varphi'(L')$ intersect locally cleanly, which means that there are finite covers $L = \bigcup_{i=1}^k U_i$, $L' = \bigcup_{i=1}^l U_i'$ such that φ resp. φ' restrict to an embedding on each U_i resp. U_i', and $\varphi(U_i), \varphi'(U_j')$ intersect cleanly for all i,j.

Proposition A.0.6. *If u, γ, γ' satisfy* (A.1)*, then u extends continuously to* 0.

Sketch proof of Proposition A.0.6. The first part of the proof of [AbbHo, Theorem 7.3.1] yields a uniform gradient bound on u in cylindrical coordinates near the puncture. We must make a minor modification due to the fact that the Lagrangians defining our boundary conditions are immersed, not embedded: Recall that the uniform gradient bound in cylindrical coordinates is established in [AbbHo] by assuming that there is a sequence $((s_k,t_k)) \subset (-\infty,0]\times[0,\frac{1}{2}]$ so that $\lim_{k\to\infty}|du(s_k,t_k)| = \infty$, which necessarily has $s_k \to -\infty$. Rescaling at the points (s_k,t_k) yields a sequence of maps that converges in $\mathcal{C}^\infty_{\mathrm{loc}}$ to a nonconstant map on either $\mathbb{R}^2$ or $\pm\mathbb{H}$, which contradicts the finiteness of the energy. To adapt this proof to our situation, let δ be a Lebesgue number for $L = \bigcup_{i=1}^k U_i$ and $L' = \bigcup_{i=1}^l U_i'$. That is, if A is a subset of L (resp. of L') with $\operatorname{diam} A \le \delta$, then $A \subset U_i$ (resp. $A\subset U_i'$) for some i. Now rescale at the points (s_k,t_k) as in [AbbHo], but restrict the resulting maps to the intersection of $B(0,\frac{1}{4}\delta)$ with their domain. The gradient bound on these rescaled maps and our choice of δ allows us to pass to a subsequence so that for some i,j, all the rescaled maps have boundary values in $\pi(U_i)$ or $\pi'(U'_j)$. A further subsequence converges in $\mathcal{C}^\infty_{\mathrm{loc}}$, so we get a contradiction and therefore a uniform bound on $|\nabla u|$ in cylindrical coordinates.

The analogue of Lemma 2.1.3 holds in this setting; the proof is the same as for Lemma 2.1.3 but simpler. As in the first paragraph, some care must be taken with the immersed Lagrangians.

The analogue of Lemma 2.1.8 holds in this setting, though the proof must be modified. Specifically, the domains U_0, U_1, U_2, U_3 used in the proof of that lemma must be replaced by the domain $\overline{B}(0,1) \cap \mathbb{H}$.

A slight modification of the proof of Theorem 2.1.2 establishes Proposition A.0.6. $\square$

Appendix B

Examples of figure eight bubbles (in collaboration with Felix Schmäschke)

In this section we provide some examples of figure eight bubbles. Our first, previously known, examples show that classical holomorphic discs and holomorphic strips give rise to figure eight bubbles, which naturally appear in the strip shrinking limit of Theorem 3.3.1 in case M_1 or both of M_0, M_2 are points. Of course, in that case, strip shrinking is not needed to identify the respective moduli spaces.

Example B.0.7. Let M_0 and M_2 each be a point, let M_1 be any symplectic manifold that is either compact or satisfies the boundedness assumptions of Remark 3.3.4, and let $L, L' \subset M_1$ be any two compact Lagrangian submanifolds. Then a J_1-holomorphic strip $u_1 \colon [-1,1] \times \mathbb{R} \to M_1$ with boundary conditions

$$u_1(-1,t) \in L, \qquad u_1(1,t) \in L' \qquad \forall t \in \mathbb{R}$$

gives rise to a figure eight bubble in the sense of Definition 3.2.5 by setting $u_0 :=$ const $=: u_2$. Such bubbles are generally sheet-switching, unless u_1 is a self-connecting Floer trajectory.

Here the correspondences $\{\mathrm{pt}\} \times L$ and $L' \times \{\mathrm{pt}\}$ have immersed composition if and only if the Lagrangians intersect transversely in M_1. This bubble type in fact appears in the strip shrinking that relates the quilted Floer trajectories for $(\{\mathrm{pt}\}, \{\mathrm{pt}\} \times L, L' \times \{\mathrm{pt}\}, \{\mathrm{pt}\})$ and $(\{\mathrm{pt}\}, (\{\mathrm{pt}\} \times L) \circ (L' \times \{\mathrm{pt}\}), \{\mathrm{pt}\})$. The first are easily identified with the Floer strips for (L, L'). The latter are pairs of strips in M_0 and M_2, hence there only is one constant trajectory. All nontrivial Floer trajectories result in a single figure eight bubble on this constant trajectory. This demonstrates that figure eight bubbling must be reckoned with, even when only considering isolated Floer trajectories.

All following next figure eight bubbles will be constructed as tuples of maps from the following Riemann surfaces with boundary:

$$\Sigma_0 = \{w \in \mathbb{C} \mid |w+1| \le 1\} \setminus \{0\}, \qquad \Sigma_2 = \{w \in \mathbb{C} \mid |w-1| \le 1\} \setminus \{0\},$$
$$\Sigma_1 = \{w \in \mathbb{C} \mid |w+1| \ge 1, |w-1| \ge 1\} \setminus \{0\}.$$

Each of these surfaces is equipped with the complex structure induced by the inclusion into $\mathbb{C}$, and we will ensure that the maps on Σ_1 extend smoothly to $\infty \in \mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}$. The coincidence of boundary components in $\mathbb{C}$ then induces seams between the surfaces and thus defines a quilted surface with total space $\mathbb{CP}^1 \setminus \{0\}$. It can be identified, by a biholomorphism $\mathbb{CP}^1 \setminus \{0\} \cong \mathbb{C} \setminus \{\infty\}$, with the quilted surface underlying the figure eight bubbles in Definition 3.2.5.

Example B.0.8. Let M_1 be a point, let M_0, M_2 be any two symplectic manifolds that are either compact or satisfy our boundedness assumptions, and let $L \subset M_0$ and $L' \subset M_2$ be any two compact Lagrangian submanifolds. Given two punctured holomorphic discs $u_0 \colon \Sigma_0 \to M_0$, $u_2 \colon \Sigma_2 \to M_2$ with $u_0(\partial\Sigma_0) \subset L$, $u_2(\partial\Sigma_2) \subset L'$, we obtain a figure eight by setting $u_1 := \text{const}$.

Here the correspondences $L \times \{\text{pt}\}$ and $\{\text{pt}\} \times L'$ always have embedded composition $L \times L' \subset M_0 \times M_2$. The singularity in these figure eight bubble is already removed by our construction, and for other bubbles of this type can be removed by the standard result for punctured disks, yielding a pair of disk bubbles on L and L'. These could occur in the strip shrinking that, for further Lagrangians $L_0 \subset M_0$ and $L_2 \subset M_2$, relates the quilted Floer trajectories for $(L_0, L \times \{\text{pt}\}, \{\text{pt}\} \times L', L_2)$ and $(L_0, L \times L', L_2)$. However, these moduli spaces also have an elementary identification, so this type of figure eight bubbling just is an expression of the fact that the moduli space has boundary components where disk bubbles appear at the same $\mathbb{R}$ coordinate on different seams. Actual boundary, rather than corners, correspond to one of the disk bubbles being constant.

For the final, more nontrivial example, $\mathbb{CP}^n$ will denote the complex projective space equipped with its standard complex structure and with Kähler form $\omega_{\mathbb{CP}^n}$ associated to the Fubini-Study metric.

Example B.0.9. Consider the S^1-action on $\mathbb{CP}^3$ given by $u * [z_0 : z_1 : z_2 : z_3] := [uz_0 : uz_1 : u^{-1}z_2 : u^{-1}z_3]$ for any $u \in \{z \in \mathbb{C} : |z| = 1\} \cong S^1$. This is a Hamiltonian S^1-action with Hamiltonian

$$\mu([z_0 : z_1 : z_2 : z_3]) := \frac{1}{2}\frac{|z_0|^2 + |z_1|^2 - |z_2|^2 - |z_3|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2}.$$

Symplectic reduction at regular values generally gives rise to Lagrangian correspondences, see [WeWo4, Example 2.0.2(e)]. In this case, reduction at 0 yields a Lagrangian correspondence between $M_0 = M_2 := \mathbb{CP}^3$ and $M_1 := \mathbb{CP}^1 \times \mathbb{CP}^1$, equipped with the symplectic structure $\omega := \omega_{\mathbb{CP}^1} \oplus \omega_{\mathbb{CP}^1}$. Indeed, the level set of the moment map is

$$\mu^{-1}(0) = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \;\middle|\; |z_0|^2 + |z_1|^2 = |z_2|^2 + |z_3|^2 \right\}$$

and the quotient map $\pi \colon \mu^{-1}(0) \to M_0 /\!/ S^1 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$ is given by $[z_0 : z_1 : z_2 : z_3] \mapsto ([z_0 : z_1], [z_2 : z_3])$. With the inclusion $\iota \colon \mu^{-1}(0) \to \mathbb{CP}^3$, this gives rise to Lagrangian submanifolds $L_{01} := (\iota \times \pi)(\mu^{-1}(0)) \subset M_0^- \times M_1$ and $L_{12} := (\pi \times \iota)(\mu^{-1}(0)) \subset M_1^- \times M_2$. Both are diffeomorphic to $S^3 \times S^2$, hence simply connected; therefore L_{01} and L_{12} are monotone, with the same monotonicity constant as $M_0^- \times M_1$ resp. $M_1^- \times M_2$. The geometric composition is

$$L_{01} \circ L_{12} \;=\; \{(Z_0, Z_2) \in \mu^{-1}(0) \times \mu^{-1}(0) \mid \pi(Z_0) = \pi(Z_2)\} \;\subset\; \mathbb{CP}^3 \times \mathbb{CP}^3,$$

and is embedded since π is a surjection and determines $\pi(Z_0) = \pi(Z_2) \in M_1$ uniquely.

Now a general idea for constructing figure eight bubbles applies to this case. The holomorphic map $\mathbb{C} \to \mathbb{C}^4, w \mapsto (w-1, w+1, w^2-1, 1)$ induces holomorphic maps to both $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^3$, and on the seams $\{|w \pm 1| = 1\} \subset \mathbb{C}$ the latter takes values in $\mu^{-1}(0)$.

Hence the following triple (u_0, u_1, u_2) defines a nonconstant figure eight bubble,

$$\begin{aligned}
u_0 \colon \Sigma_0 &\to \mathbb{CP}^3, & w &\mapsto [w-1 : w+1 : w^2-1 : 1], \\
u_1 \colon \Sigma_1 &\to \mathbb{CP}^1 \times \mathbb{CP}^1, & w &\mapsto \big([w-1 : w+1], [w^2-1 : 1]\big), \\
u_2 \colon \Sigma_2 &\to \mathbb{CP}^3, & w &\mapsto [w-1 : w+1 : w^2-1 : 1].
\end{aligned}$$

Note here that u_1 extends continuously to $\infty \in \mathbb{CP}^1$ since $[w-1 : w+1] \to [1 : 1]$ and $[w^2-1 : 1] \to [1 : 0]$ as $w \to \infty$. Moreover, all maps extend smoothly to $0 \in \mathbb{C}$ — an example of a removable singularity.

This figure eight bubble could occur in the strip shrinking that, for further Lagrangians $L_0, L_2 \subset \mathbb{CP}^3$ relates the quilted Floer trajectories for $\big(L_0, L_{01}, L_{12}, L_2\big)$ and $\big(L_0, L_{01} \circ L_{12}, L_2\big)$. In particular, if both L_0, L_2 are monotone and so-called "transverse lifts" of Lagrangians $\ell_0, \ell_2 \subset \mathbb{CP}^1 \times \mathbb{CP}^1$, i.e. $L_i \pitchfork \mu^{-1}(0)$ and $\pi \colon L_i \cap \mu^{-1}(0) \to \ell_i$ is bijective, then the above figure eight bubble could be an obstruction to the identification of the Floer homologies $HF(\ell_0, \ell_2) \cong HF(L_0, L_{01}, L_{12}, L_2)$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $HF(L_0 \times L_2, L_{01} \circ L_{12})$ in $(\mathbb{CP}^3)^- \times \mathbb{CP}^3$.

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